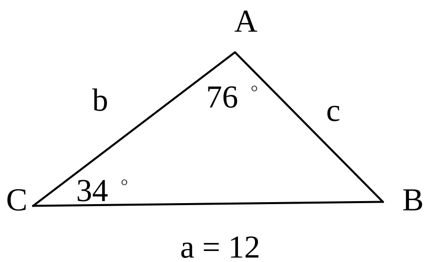
Unit 6 Vectors

Law of Sines – oblique triangles

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

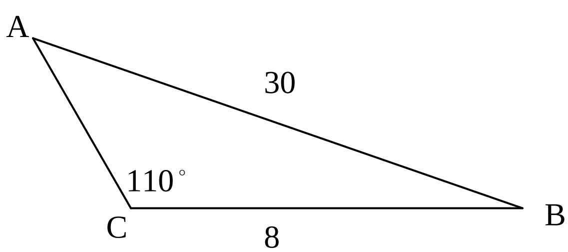
1) Two angles and any side (AAS or ASA)
2) Two sides and an angle opposite one of them.

Ex:



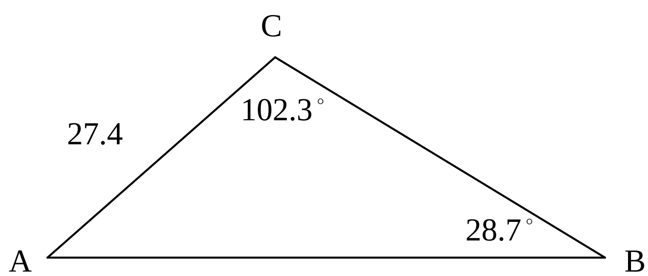
Find b and c.

Ex:



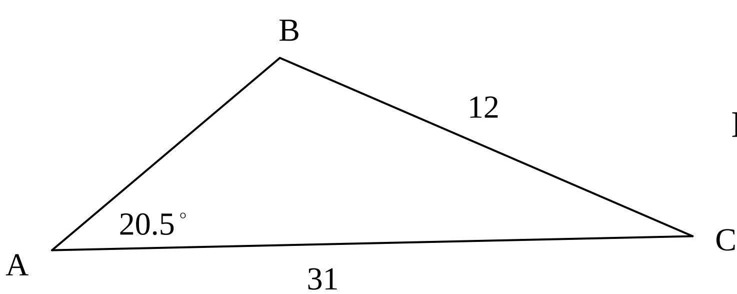
Find angle A & B

Ex:



Ambiguous case: Finding the remaining sides and angles.

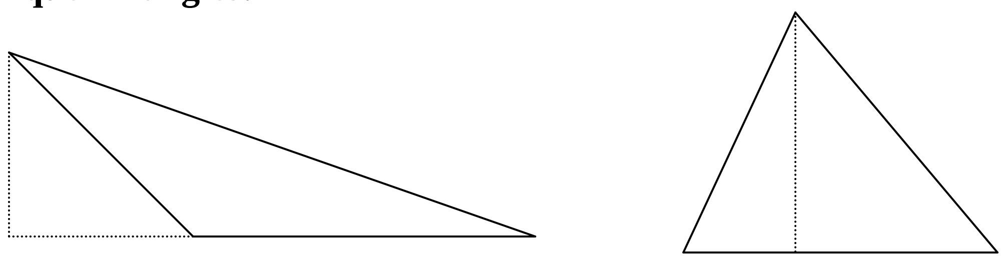
Ex:



Ex: c = 29, b = 46, and $\angle C = 31^\circ$

Ex: a = 15, b = 25, and A = 85°. Find the remaining angles and sides.

Area of Oblique Triangles:



Area = ½ ab sin C = ½ bc sin A = ½ ac sin B

Ex: Find the area of the triangle with the indicated values

A = 105°, c = 8, and b = 12

Law of Cosines:

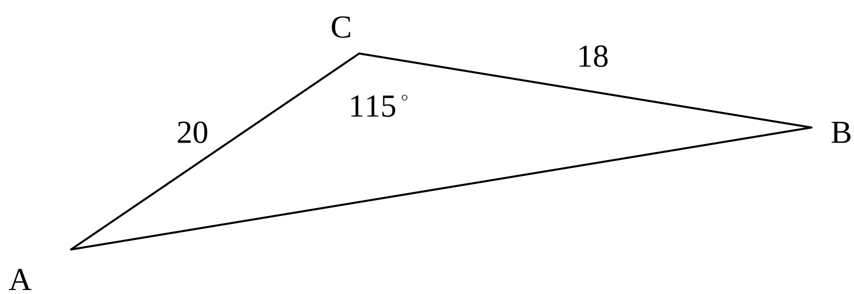
$a^2 = b^2 + c^2 - 2bc \cos A$
$b^2 = a^2 + c^2 - 2ac \cos B$
$c^2 = a^2 + b^2 - 2ab \cos C$

1) Three sides (SSS)
2) Two sides and an angle in between (SAS)

Ex: a = 6.2, b = 12.4, and c = 8.1

Ex: $\angle B = 55^\circ$, b = 13, and a = 19

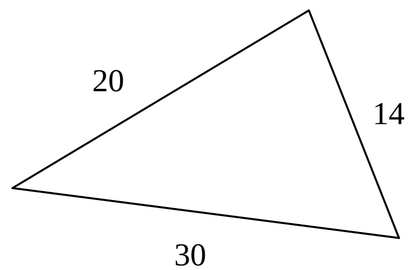
Ex:



Show Proof:

Heron's Formula:

Find the area of the triangle.



Vectors in the Plane:

Many quantities such as length, mass, volume can be specified by a single value. (scalars)

Others such as velocity, force, torque, and displacement require a magnitude and a direction. (vectors)

Geometrically a **vector** is a directed line segment with a certain length and direction.

Vector – the set of all equivalent line segments.

Ex: directed line segment JG has an initial point J(tail) and a terminal point G(head).

Length of JG = ||JG||

vector **w** is the set of all vectors that are equivalent to JG

Must have same
slope & length
same direction

Ex: Let **u** be a directed line segment from (0,0) to (3,2) and v be directed line segment from (1,2) to (4,4). Show **u** = **v**.

Find the length:

Find the slope:

Component Form of a Vector:

A vector in standard position is usually the most convenient way to write the vector.

Standard Position – initial point (tail) is (0,0).
A vector in standard position is denoted by its' terminal point

$$v = \langle v_1, v_2 \rangle \quad \text{component form of a vector}$$

To put into component form:

If the initial point is $\langle p_1, p_2 \rangle$ and the terminal point is $\langle q_1, q_2 \rangle$ then:

$$\langle v_1, v_2 \rangle = \langle q_1 - p_1, q_2 - p_2 \rangle$$

length or **magnitude** of v is: $\sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2} = \sqrt{v_1^2 + v_2^2}$

Zero vector has both initial and terminal points at (0,0).

Ex: Finding component form and length of v with initial point (3,-6) and terminal point (-4,2).

Vector Operations:

Basic Operations

1) scalar multiplication
2) vector addition

Scalar Multiplication:

If you multiply a constant k times a vector, the product is |k| times as long as **v**. If k is positive, it has the same direction and if k is negative, it goes in the opposite direction.

Geometric representation of scalar multiplication:

Algebraic: $kv = k < v_1, v_2 > \; = < kv_1, kv_2 >$

Ex: let **v** = < -7, 8 >, find 2**v**:

Ex: let **u** = < -3, 4 > find **-u**

Vector Addition:

Geometric:

Put tail of **v** to head of **u**

Where is **u** + **v**?

Initial of **u** drawn to head of **v**.

Addition is commutative, associative, and distributive

Algebraically: **u** = < 3, -6 > **v** = < -5, 2 >

u + **v** =

Vector Subtraction **u** – **v** . Think of this as **u** + (-**v**)

Geometric:

Algebraic $\mathbf{u} - \mathbf{v} = \langle u_1 - v_1, u_2 - v_2 \rangle$

Ex: Let $\mathbf{m} = \langle -1, 5 \rangle$, $\mathbf{n} = \langle 4, -2 \rangle$

1. Find $-3\mathbf{m}$
2. Find $\mathbf{n} - \mathbf{m}$
3. Find $-2\mathbf{m} + \mathbf{n}$

Unit Vector:

The unit vector **u** has length 1 and the same direction as vector **v**.

To find: divide **v** by its' length $\frac{v}{\| v \|}$.

Ex: $\mathbf{v} = \langle -7, 8 \rangle$

Unit vector:

Standard Unit Vectors $\langle 1, 0 \rangle$ and $\langle 0, 1 \rangle$

We always use $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$

$$\begin{aligned} v &= \langle v_1, v_2 \rangle \\ &= v_1 \langle 1, 0 \rangle + v_2 \langle 0, 1 \rangle \\ &= v_1\mathbf{i} + v_2\mathbf{j} \end{aligned}$$

$v = v_1\mathbf{i} + v_2\mathbf{j}$ is called a <u>linear combination</u>.

To write Linear Combination:

1) Write the vector in component form
2) Use **i** and **j** to write the equation

Ex: **v** = vector from (1, 4) to (-3, 6), write as a linear combination

Ex: **u** = -2**i** – 6**j** and v = -4**i** + 2**j**

Find 3**u** + 2**v**:

You could solve by converting back to component form but it is not necessary.

Ex: unit vector **w** = -5**i** – 3**j**

Show **u** + **v** graphically with a vector from (1,3) to (-3,-4) and a vector from (2,-2) to (4,-5). Move to standard position first.

Algebraically:

Direction Angles:

If u is a unit vector and θ is the angle (counter-clockwise) from the x-axis, then **u** = < cos θ, sin θ > because its' terminal point is on the unit circle.

θ = direction angle

unit vector = < cos θ, sin θ >

$$\frac{v}{\| v \|} = \langle \cos\theta, \sin\theta \rangle$$

v = ||v|| < cos θ, sin θ > or **v** = ||v|| [(cos θ)**i** +(sin θ)**j**]

To find Directional Angles:

1) Put the vector in compound form or as a linear combination
2) Find tan θ
3) Find θ

Ex: Find the direction angle for:

A: **v** = 6**i** + 6**j** or < 6, 6 >

B: **v** = 2**i** – 5**j** or < 2, -5 >

Dot Product – different from vector addition and scalar multiplication because in those you get a vector answer & in this you get a scalar answer.

Definition: Dot product of **u** = < u_1, u_2 > and **v** = < v_1, v_2 >

$$\mathbf{u} \bullet \mathbf{v} = u_1v_1 + u_2v_2$$

Properties:

1) $\mathbf{u} \bullet \mathbf{v} = \mathbf{v} \bullet \mathbf{u}$
2) $0 \bullet \mathbf{v} = 0$
3) $\mathbf{u} \bullet (\mathbf{v} + \mathbf{w}) = \mathbf{u} \bullet \mathbf{v} + \mathbf{u} \bullet \mathbf{w}$
4) $\mathbf{v} \bullet \mathbf{v} = \|v\|^2$
5) $k(\mathbf{u} \bullet \mathbf{v}) = k\mathbf{u} \bullet \mathbf{v} = \mathbf{u} \bullet k\mathbf{v}$

Ex: < 6,8 > • < 1,2 > =
Ex: < 3,-5 > • < 3,2 > =
Ex: < 0, 4 > • < 2,1 > =

Using Properties

Ex: Let **u** = < 1,2 > , **v** = < 3,4 > and **w** = < -1, 2 >

Find **u**(**v** • **w**) =

Find **u** • 3**w** =

Dot Product & Length

The dot product of **u** with itself is 7. What is the magnitude of **u**?

The Angle Between two non-zero vectors:

If θ is the angle between 2 non-zero vectors when **u** & **v** are in standard form $0 \le \theta \le \pi$:

$$\cos\theta = \frac{u \bullet v}{\| u \| \| v \|}$$

Find the angle between **u** = < 3,2 > and **v** = < 1,4 >

Note: can also be rewritten as **u** • **v** **=** ||u||||v|| cos θ

Orthogonal Vectors:

Definition: vectors are orthogonal if their dot products are 0.

If $\mathbf{u} \cdot \mathbf{v} = \mathbf{0}$**,** then **u** and **v** are orthogonal.
Orthogonal basically means perpendicular.

Ex: Show that vectors **u** = < 2, -3 > and **v** = < 6,4 > are orthogonal.

Ex: Find the measure of the angle ABC where A = (4, 3), B = (1, -1) and C = (6 , -4).

Proof of Properties:

Prove $\mathbf{v} \cdot \mathbf{v} = ||\mathbf{v}||^2$

Prove $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

Prove $(r\mathbf{u}) \cdot \mathbf{v} = r(\mathbf{u} \cdot \mathbf{v})$

DeMoivre's Theorem:

Graphing Complex Numbers

Absolute value of a complex number: a + bi = | a + bi | = $\sqrt{a^2+b^2}$
Distance between (0,0) and (a,b).

Ex: z = -3 + 4i; find the absolute value

Polar Form of a Complex Number:

Polar form of z = a + bi is z = r(cos θ + $i$sin θ)
Where a = rcos θ
b = rsin θ
r = $\sqrt{a^2+b^2}$ r = modulus
tan θ = $\frac{b}{a}$ θ = argument

Ex: Write z = -3 + $\sqrt{3}\,i$ in trig form (polar form)

1. Graph 2) Find r 3) Find θ

4) Polar Form:

Complex Form of a Polar:

Ex: Write $z = 6(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3})$ in standard form

Multiplication and Division of Complex Numbers

Let $z_1 = r_1(\cos\theta + i\sin\theta)$ and $z_2 = r_2(\cos\theta + i\sin\theta)$

Product $z_1 \bullet z_2 = r_1 r_2 (\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$

Quotient $\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$

Ex: Find the product z_1z_2 if $z_1 = 3(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})$

$z_2 = 4(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6})$

Ex: Find the quotient $\frac{z_1}{z_2} = \frac{3}{4}(\cos(\frac{2\pi}{3} - \frac{7\pi}{6}) + i\sin(\frac{2\pi}{3} - \frac{7\pi}{6}))$

Ex: if $z_1 = 3(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})$ and $z_2 = 4(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6})$,

find z_1z_2 and $\frac{z_1}{z_2}$.

Powers of Complex Numbers:

DeMoivre's Theorem → if $z = r(\cos\theta + i\sin\theta)$ is a complex number and n is a positive integer, then $z^n = r^n(\cos n\theta + i\sin n\theta)$

Ex: Find $(-2 - 2i\sqrt{3}\ )^8$

1. Convert to trig form (polar)

2. Use DeMoivre's Theorem

Ex: Find $(-4 - 2i\ )^6$

Ex: $\dfrac{5}{2+3i}$

Ex: $(5+ i)(3 - 2i)$

Definition of n^th root of a Complex Number.

$u = a + bi$ is the n^{th} root of a complex number z if $z = u^n = (a + bi)^n$.

$$\sqrt[n]{z} = a + bi$$

For a positive integer n, the complex number $z = r(\cos\theta + i\sin\theta)$ has exactly n distinct n^{th} roots given by:

$$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta + 2\pi k}{n} + i\sin\frac{\theta + 2\pi k}{n}\right)$$ where k = 0,1,2,3…n – 1

Ex: Find the cube root of $-8i$.