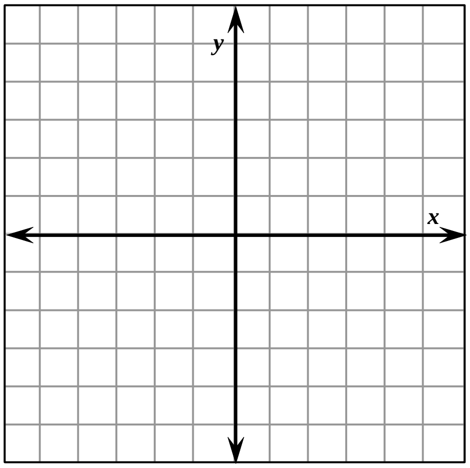
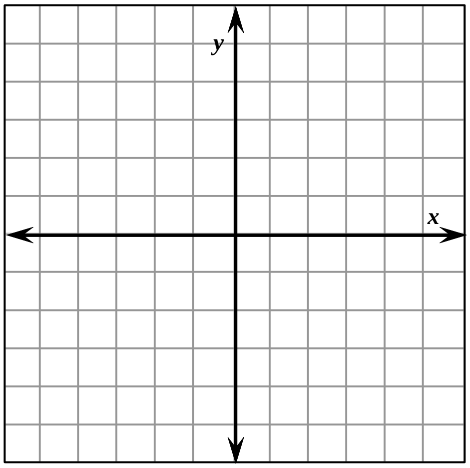
Exponential Function:
Has the form $y = a^x$, where $a > 0,\ a \neq 1$ and x is any real number.

Graph $y = e^x$,



Graph $y = \ln x$



The Natural Base *e* (Euler's number):
An irrational number, symbolized by the letter *e*, appears as the base in many applied exponential functions. This irrational number is approximately equal to 2.72. More accurately, $e = 2.71828..$. The number *e* is called the natural base.

The Natural Logarithm function has the following properties:

1. The domain is $(0, \infty)$ and the range is $(-\infty, \infty)$

2. The function is continuous, increasing, and one-to-one.

3. The graph is concave downward.

Logarithm Properties: If a and b positive numbers and n is rational, then the following are true.

1. $\ln(1) = 0$

2. $\ln(ab) = \ln a + \ln b$

3. $\ln(a^n) = n \ln a$

4. $\ln\left(\dfrac{a}{b}\right) = \ln a - \ln b$

Definition of the Natural Logarithm Function: $\ln x = \int_1^x \frac{1}{t}\,dt,\ x > 0$

Expanding Logarithms:

Ex: $\ln \frac{3x^2}{5} =$

Ex: $\ln \frac{(2x+1)^3}{\sqrt{x^2-1}} =$

Derivative of the Natural Logarithmic Function

$\frac{d}{dx}[\ln x] = \frac{1}{x},\ x > 0$

$\frac{d}{dx}[\ln u] = \frac{1}{u}\frac{du}{dx} = \frac{u'}{u},\ u > 0$

Find the derivative of each of the following:

Ex: $y = \ln(5x)$

Ex: $y = \ln|\sin x|$

Ex: $y = 3x\ln(x)$

Ex: $y = \ln\sqrt{x-3}$

Ex: $y = \ln \frac{x(x^2-5)}{\sqrt{x+6}}$

Ex: $y = \ln(\ln x)$

Logarithmic Differentiation:

Ex: $y = \frac{(x-2)^2}{\sqrt{x^2+4}}$

Ex: Find all relative extrema and inflection points for $y = \frac{x^2}{2} - \ln x$

Integration – The Natural Logarithm

$$\int \frac{1}{x}\,dx = \ln|x| + C \qquad\qquad \int \frac{1}{u}\,du = \ln|u| + C$$

Ex: $\int \frac{4}{x}\, dx$

Ex: $\int \frac{1}{2x+5}\, dx$

Ex: $\int \frac{3x}{x^2+1}\, dx$

Ex: $\int \frac{20x^4}{4x^5+3}\, dx$

Ex: $\int \frac{\sec^2\theta}{\tan\theta}\, d\theta$

Ex: $\int \frac{x+1}{x^2+2x}\, dx$

Ex: $\int \frac{2x}{(x+1)^2}\, dx$

Ex: $\int \sec x\, dx$

Ex: $\int_1^4 \frac{x}{x^2+1}\, dx$

Find the average value.

Long Division Before Integrating Ex: $\int \frac{x^2+x+1}{x^2+1}\, dx$

Ex: $\int_0^{\pi/4} \sqrt{1+\tan^2 x}\, dx$

Warm Up

1. Find the derivative of $y = \dfrac{(x+3)}{(3x^2 - 4x)^{\frac{2}{3}}}$ (Do not simplify)

2. Find the integral of $\displaystyle\int \frac{4x^3 - 8x^2 - 3x + 3}{2x+1}\,dx$

Inverse Functions:
Let f and g be two functions such that $f(g(x)) = x$ for every x in the domain of g and $g(f(x)) = x$ for every x in the domain of f.

The function g is the inverse of the function f, and is denoted by f^{-1} (read "f-inverse").

Thus, $f\left(f^{-1}(x)\right) = x$ and $f^{-1}\left(f(x)\right) = x$
The domain of f is equal to the range of f^{-1}, and vice versa.

The graph of an inverse is the reflection of the original function over the line y = x.

I. Verify that the following are inverses: Show that $f\left(f^{-1}(x)\right) = x$ and $f^{-1}\left(f(x)\right) = x$.

1. $f(x) = 2x + 3$

   $g(x) = \dfrac{x-3}{2}$

2. $f(x) = \sqrt{3x+5}$

   $g(x) = \dfrac{x^2 - 5}{3} \quad x \geq 0$

1. To have an ***inverse function***, a function must be one-to-one, which means no two elements in the domain correspond to the same element in the range of f.

You can use the horizontal line test to determine if a function is one-to-one.

2. If f is strictly monotonic on its entire domain, then it is one-to-one and therefore has an inverse function.

A function is monotonic if it is either strictly increasing or decreasing on its entire domain.

Do the following functions have an inverse function? Use the derivative .

Ex: $f(x) = \sqrt{x+2}$ $(-2, \infty)$

Ex: $g(x) = -\dfrac{1}{6}x^3 - 5x + 4$ $[0, \infty)$

Ex: $h(x) = \sin x$ $\left(\pi, \dfrac{3\pi}{2}\right)$

Ex: $p(x) = \dfrac{1}{x}$

Discuss $f(x) = x^2$ and the $g(x) = \sqrt{x}$.

Ex: $f(x) = \sqrt{x^3 + 1}$ Find $\left(f^{-1}\right)'(3) =$

Ex: $f(x) = \ln x$ Find $\left(f^{-1}\right)'(1) =$

Inverses Cont:

Let $f(x) = \sqrt{x+2}$ find the inverse. Label the inverse as a function named *g*.

What is the slope of the tangent line to *f* at x = 7?

We know that $f(x) = \ln x$ and $g(x) = e^x$ are inverses. Namely that $\ln\left(e^x\right) = x$ and $e^{\ln x} = x$.

Ex: A girl invests $500 in a bank with an interest of 3 % compounding continuously. How long before her investment doubles?

Ex: Solve $9 = e^{2x+1}$

Ex: Solve $\ln(x-5) = 4$

Ex: Solve $\ln(2x+3) = \frac{3}{2}$

Derivative of the Natural Exponential Function

$$\frac{d}{dx}\left[e^x\right] = e^x \qquad \frac{d}{dx}\left[e^u\right] = e^u \frac{du}{dx}$$

Find the derivative of each of the following:

Ex: $y = e^{3x+1}$

Ex: $y = e^{\frac{1}{2}x - 5}$

Ex: $y = e^{\frac{-3}{x}}$

Ex: $y = e^{\cos(x)}$

Ex: $y = x^2 e^{2x}$

Integration – The Natural Exponential Function

$$\int e^x \, dx = e^x + C \qquad \int e^u \, dx = e^u + C$$

Ex: $\int e^{2x+5} \, dx$

Ex: $\int e^{\frac{1}{3}x-1} \, dx$

Ex: $\int 7x^2 e^{x^3} \, dx$

Ex: $\int \sec^2(x)\, e^{\tan x} \, dx$

Ex: $\int \frac{e^x}{2-e^x} \, dx$

Ex: $\int e^{\sec(2x)} \sec(2x)\tan(2x) \, dx$

Differentation & Integration – Bases other than e

Definition of Exponential Function to Base a: If a is a positive real number ($a \neq 1$) and x is any number, then the **exponential function to the base a** is denoted by a^x and is defined as $a^x = e^{(\ln a)x}$, if a = 1, then y = 1 (the constant function).

Definition of Logarithmic Function to Base a: If a is a positive real number ($a \neq 1$) and x is any number, then the **logarithmic function to the base a** is denoted by $\log_a x$ and is defined as $\log_a x \log_a x = \dfrac{\ln x}{\ln a}$.

Derivative for Bases Other than e.

$$\frac{d}{dx}\left[a^x\right] = (\ln a)a^x \qquad \frac{d}{dx}\left[a^x\right] = (\ln a)a^u\,\frac{du}{dx}$$

$$\frac{d}{dx}\left[\log_a x\right] = \frac{1}{(\ln a)x} \qquad \frac{d}{dx}\left[\log_a u\right] = \frac{1}{(\ln a)u}\frac{du}{dx}$$

Ex: $f(x) = 4^x$ Ex: $f(x) = 4^{2x-3}$ Ex: $f(x) = 4^{x^3}$ Ex: $f(x) = \log(\cos x)$

Ex: $f(x) = \log_3 \dfrac{\sqrt{x}}{x+3}$ Ex: $f(x) = 3^{-x}\tan\left(x^2\right)$ Ex: $f(x) = \log_a a^{\sin x}$

$$\int a^x\,dx = \left(\frac{1}{\ln a}\right)a^x + C \qquad \int a^u\,du = \left(\frac{1}{\ln a}\right)a^u + C$$

Ex: $\int 5^{-x+3}\,dx$ Ex: $\int 3^{2x+5}\,dx$ Ex: $\int \dfrac{2^{3x}}{4+2^{3x}}\,dx$

Power Rule versus Logarithmic Differentiation

Show how they both work with a polynomial: $f(x) = x^n$

Differential Equations: Growth & Decay

A differential equation in x and y is an equation that involves x, y, and derivatives of y. The strategy is to rewrite the equation so that each variable occurs on only one side of the equation. This is called separation of variables.

1. $\dfrac{dy}{dx} + \dfrac{4x}{y^2} = 0$

2. $\dfrac{dy}{dt} = 8\sqrt{y}$

General Solution:

If $y(1) = 2$

If $y(0) = 16$

3. $\dfrac{dy}{d\theta} = \tan\theta$

4. $\ln y \; y' - ty = 0$

General Solution:

If $y\left(\dfrac{\pi}{4}\right) = \ln\sqrt{2} + 3$

5. $\frac{dy}{dx} = x(y+1)$

6. $y\ y' = xe^{-y^2}$

7. $y' + 2y = 0$

General Solution:
If $y(0) = 3$

Rate of Growth of a population: The following describes a population whose growth will be exponential.

$$\frac{dP}{dt} = kp$$

This population grows at a rate proportional to the amount present at any time (Assume population $\neq$ 0)

$$\frac{dP}{dt} = kp$$

Let $P(0) = P_0$

Differential Equations – Separation of Variables

If y is a differentiable function of t such that y > 0 and $y' = ky$, for some constant k, then $y = Ce^{kt}$.

C is the initial value of y, and k is the proportionality constant. Exponential growth occurs when k > 0, and decay occurs when k < 0.

Ex: Carbon (14) has a half-life of 5,730 years. If the initial value of y is 32,000. How much of the substance will be left after 100 years, 42,000 years?

Ex: The rate of change of *N(t)* number of bears in a population is proportional to $1200 - N(t)$, $t \geq 0$ is time in years. Use that *N(4)= 600.*

A) Write the differential equation.

B) Solve the equation.

C) What is *N(8)*?

D) Find the limit as t goes to infinity of *N(t).*

Ex: Suppose an experimental population of flies increases according to the exponential of growth. There were 100 flies after the second day of the experiment and 300 flies after the fourth day. Approximately how many flies were in the original population?

Ex: A certain type of bacteria increases continuously at a rate proportional to the number present at time t. If there are 500 at a given time and 1000 2 hours later, how many hours will it take for there to be 2500?

Ex: Newton's Law of Cooling states that the rate of change in the temperature of an object is proportional to the difference between the object's temperature and the temperature of the surrounding medium. Let y represent the temperature of an object in a room whose temperature is kept at a constant 70 degrees. If the objects cools from 120 degrees to 100 degrees in 15 minutes, how much longer will it take for its temperature to decrease to 80 degrees?

Ex: An orthogonal trajectory of a family of curves is a curve that intersects each curve of the family orthogonally, at right angles. Find the orthogonal trajectories of the family of curves $x = ky^2$, where k is an arbitrary constant.

Ex: Electro-static fields and streamlines