Unit 4 Notes

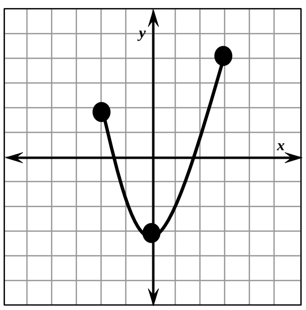
Extrema On An Interval:

Let *f* be defined on an Interval ***I*** containing c.

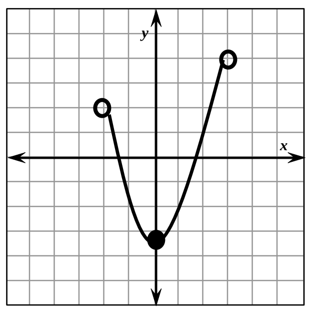
1. *f*(c) is the **minimum of *f* on *I*** if $f(c) \le f(x)$ for all x in *I*.

2. *f*(c) is the **maximum of *f* on *I*** if $f(c) \ge f(x)$ for all x in *I*.

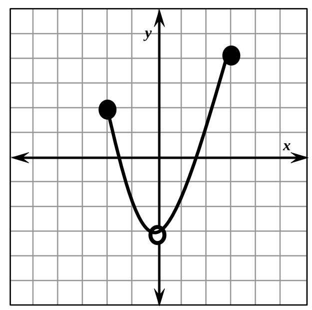
The minimum and maximum of a function on an interval are the extreme values, or extrema, of the function on the interval. The minimum and maximum are also called the **absolute minimum** and **absolute maximum** on the interval.



Max (3, 4)
Min (0, -3)



Max none
Min (0, -3)

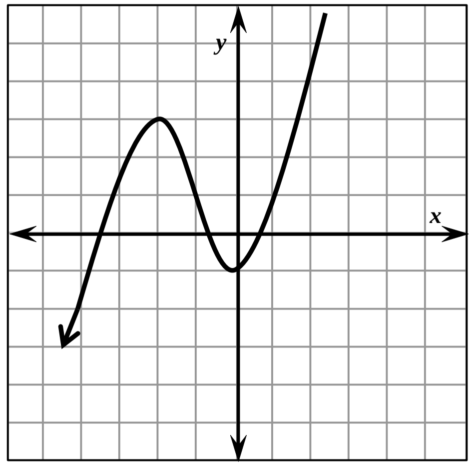


Max (3, 4)
Min none

Extreme Value Theorem: (This is an existence theorem)

If *f* is continuous on the closed interval $[a, b]$, then *f* has both a minimum and a maximum on the interval.

Label Parts:



Critical Points: Let *f* be defined at c. If $f'(c) = 0$ or if *f* is not differentiable at c, then c is a critical number of *f*.

Relative Extrema occur only at critical numbers. If f has a relative minimum or maximum at x = c, then c is a critical number.

Guidelines for Finding Extrema on a Closed Interval

1. Find the critical numbers of *f* in (a, b)
2. Evaluate *f* at each critical number in (a, b)
3. Evaluate *f* at each endpoint of [a, b]
4. The least of these values is the minimum and the greatest is the maximum.

Locate the absolute extrema of the function on the interval.

Ex 1: $y = x^3 + 6x^2 + 9x + 3; [-4, 0]$

Ex 2: $y = \dfrac{x^2}{3x - 6}; [3, 6]$

Ex 3: $y = (x)^{\frac{2}{5}}; [-1, 32]$

Ex 4: $y = \sin(x) - \cos(x); [0, \pi]$

Ex 5: $y = x^3 - 3x^2 - 3;\ (0, 3)$

Ex 6: $y = \dfrac{4}{x^2 + 2};\ (-5, -2]$

Ex 7: $y = x - \tan(x);\ \left[-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right]$

Ex 8: $y = 2\sin(x) - \cos(2x);\ [0, 2\pi]$

Rolle's Theorem:

If f is continuous on the closed interval $[a, b]$, and differentiable on the open interval (a, b)

If then $f(a) = f(b)$ then there is at least one number c in (a, b) such that $f'(c) = 0$.

(The graph has to turn around somewhere in between (a, b))

I. Find the intercepts of the given function and show that $f'(x) = 0$
at some point between the intercepts.

Ex: $f(x) = x^2 - x - 20$

Ex: $f(x) = 2x^3 - x^2 - 13x - 6$

II. Given an interval, find the c value(s) that satisfy Rolle's Theorem:

Ex: $f(x) = \dfrac{x^2 - 2x - 15}{-x + 6};\ [-3, 5]$

Ex: $f(x) = -2\sin(x);\ [-\pi, \pi]$

III. For each problem, determine if Roll's Theorem can be applied. If so, find all c values.

Ex: $f(x) = \frac{4 - x^2}{4x}$; $[-2, 2]$

Ex: $f(x) = 2\cos(x)$; $[-\pi, \pi]$

Ex: $f(x) = \tan(x)$; $[0, \pi]$

Mean Value Theorem:

If f is continuous on the closed interval $[a, b]$, and differentiable on the open interval (a, b),

then there exists a number c in (a, b) such that

$$\boxed{f'(c) = \frac{f(b) - f(a)}{b - a}}$$

The average rate of change will equal the instantaneous rate at some point on the interval.

In other words, the slope of the secant line equals the slope of the tangent line.

Ex: Cooking a hamburger: A hamburger is pulled out of the refrigerator at 50 degrees. The burger is cooked on a grill set at 500 degrees for three minutes. At some point, the temperature of the hamburger must be 150 degrees.

For the following, find the c value(s) that satisfy the Mean Value Theorem.

Ex: $f(x) = x^3 - 9x^2 + 24x - 18; [2, 4]$

Ex: $f(x) = \dfrac{x^2 - 9}{3x}; [1, 4]$

Ex: $f(x) = \dfrac{5x - 4}{x}; [1, 4]$

Ex: $f(x) = \cos(x); [0, \pi]$

Day 3:

Definition of Increasing and Decreasing Functions

A function *f* is increasing on an interval if for any two numbers a and b in the interval, a < b implies $f(a) < f(b)$.

A function *f* is decreasing on an interval if for any two numbers a and b in the interval, a < b implies $f(a) > f(b)$.

Let *f* be a function that is continuous on the closed $[a, b]$ and differentiable on the open interval (a, b).

1. If $f'(x) > 0$ for all x in (a, b), then *f* is increasing on $[a, b]$.

2. If $f'(x) < 0$ for all x in (a, b), then *f* is decreasing on $[a, b]$.

3. If $f'(x) = 0$ for all x in (a, b), then *f* is constant on $[a, b]$.

Ex 1: Find the open intervals on which $f(x) = x^3 - \dfrac{3}{2}x^2$ is increasing or decreasing. (Graph)

Ex 2: Find the open intervals on which $f(x) = \frac{x^2}{4x + 4}$ is increasing or decreasing.

Guidelines for Finding Intervals on Which a Function is Increasing or Decreasing

1. Locate the Critical Points on the interval (a, b), use these numbers to determine test intervals.

2. Determine the sign of $f'(x)$ at one test value in each of the intervals.

3. Determine whether *f* is increasing or decreasing.

The First Derivative Test:

Let c be a critical number of a function *f* that is continuous on an open interval *I* containing c.
If *f* is differentiable on the interval, except possibly at c, then $f(c)$ can be classified as follows.
1. If $f'(x)$ changes from negative to positive at c, then $f(c)$ is a relative minimum of *f*.
2. If $f'(x)$ changes from positive to negative at c, then $f(c)$ is a relative maximum of *f*.
3. If $f'(x)$ does not change sign at c, then $f(c)$ is neither a relative minimum nor a relative maximum.

Find the relative extrema of the functions below, if they exist.

Ex 3: $f(x) = -x^3 - 3x^2 - 1$

Ex 4: $f(x) = \dfrac{-2}{x^2 - 4}$

Ex 5: $f(x) = x - 2\sin(x);\ (0,\ 2\pi)$

Ex 6: $f(x) = (x^2 - 4)^{\frac{1}{3}}$

Ex 7: $f(x) = \cos^2(x) - 2\sin(x);\ (0,\ 2\pi)$

Ex 8: $f(x) = x^4 - 4x^3$

Day 4: Concavity and the Second Derivative Test

Let f be differentiable on an open interval I. The graph of f is concave upward on I if $f'(x)$ is increasing on the interval and concave downward on I if $f'(x)$ is decreasing on the interval.

Let f be a function whose second derivative exists on an open interval I.

1. If $f''(x) > 0$ for all x in I, then the graph of f is concave upward in I.

2. If $f''(x) < 0$ for all x in I, then the graph of f is concave downward in I.

Points of Inflection: If $(c,\ f(c))$ is a point of inflection of the graph of *f*, then either $f''(c) = 0$ or $f''(c)$ does not exist at x = c.

I. For each of the following find all x coordinates of all points of inflection, find all discontinuities, and find all open intervals in which the function is concave up or concave down.

Ex: $f(x) = x^3 - 3x^2 + 4$

Ex: $f(x) = x^4 - 4x^3$

Ex: $f(x) = \dfrac{1}{x-3}$

Ex: $f(x) = \dfrac{x^2+1}{x^2-4}$

Ex: $f(x) = \cos(x)$

Ex: $f(x) = 2x - \sin(x)$

What does $f''(x)$ tell us about $f'(x)$?

1. $f'(x) > 0$ and $f''(x) > 0$

3. $f'(x) < 0$ and $f''(x) < 0$

2. $f'(x) < 0$ and $f''(x) > 0$

4. $f'(x) > 0$ and $f''(x) < 0$

Match with the Graph?

A.

B.

C.

D.

Second Derivative Test:

Let *f* be a function such that $f'(x) = 0$ and the second derivative of *f* exists on an open interval containing c 1. If $f''(x) > 0$ then $f(c)$ is a relative minimum. 2. If $f''(x) < 0$ then $f(c)$ is a relative maximum. If $f'(x) = 0$, the test fails. In such cases, you can use the First Derivative Test.

Ex: $f(x) = -3x^4 + 5x^3$

Ex: $f(x) = 2x^4 - 8x + 3$

Ex: $f(x) = \dfrac{x}{x-1}$

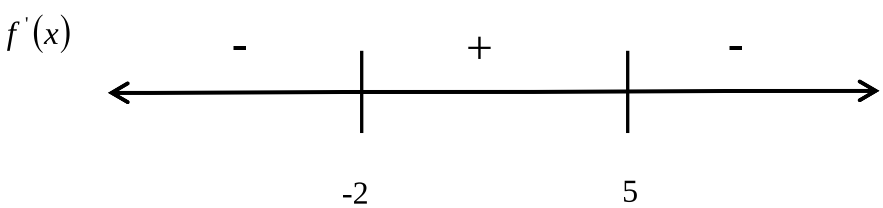
Find the relative minimum and maximum using the Second Derivative Test.

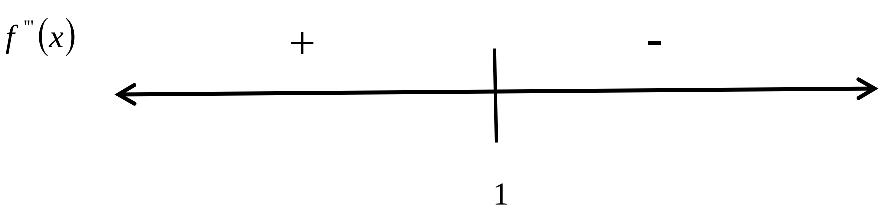
Ex: $f(x) = 2\sin(x) + \cos(2x)$

Ex: $f(x) = \begin{cases} 3 - x & x < 0 \\ 3 + 2x - x^2 & x > 0 \end{cases}$

Ex: $f(x) = \sin\left(x + \dfrac{\pi}{4}\right)$ (Sketch a graph.)

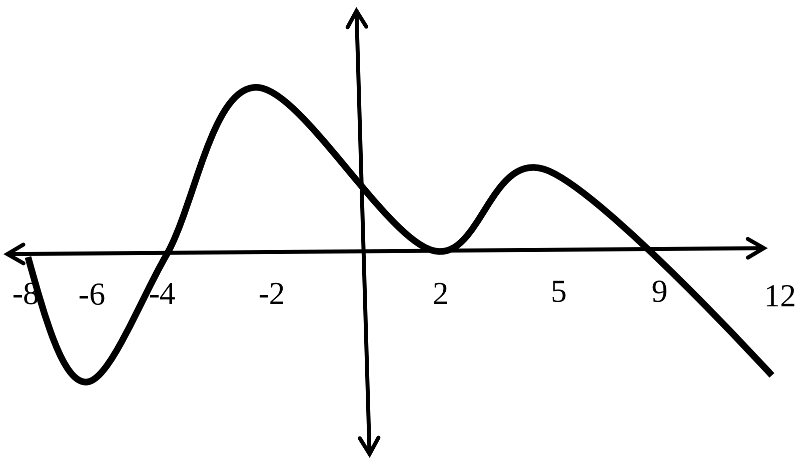
Ex: Let's look at what we know. Sketch a graph given the following information.



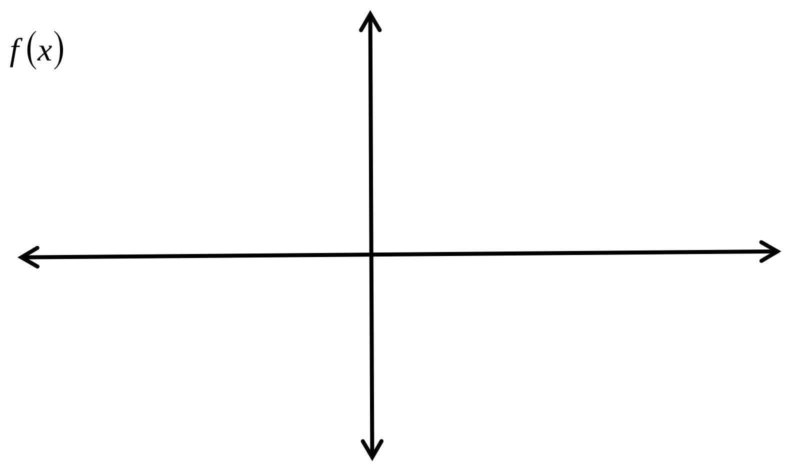


What does this tell us about *f*? Let *f*(-4) = 0 Sketch:

Let's sketch a possible graph for *f* given the following graph of $f'(x)$ on the closed interval $[-8, 12]$



Let $f(-8) = 0$



Ex: *f* is continuous on $[0, 3]$, Find the absolute extrema, point(s) of infection, and sketch.

x	0	1	2	3
$f(x)$				
$f'(x)$				
$f''(x)$				

x	$0 < x < 1$	$1 < x < 2$	$2 < x < 3$
$f(x)$			
$f'(x)$			
$f''(x)$			

Find each of the following, then sketch the graph. Show work/logic used.

NO GRAPHING CALCULATORS

1. $f(x) = -\frac{1}{3}x^3 + x^2$

A: Intercepts:

x-intercept(s) ____________________

y-intercept(s) ____________________

B: Symmetry:

y-axis: Y/N ______________

origin: Y/N ______________

C: Domain ____________________

Range ____________________

D: Vertical Asymptote(s): ______________

Horizontal Asymptote: ______________

End Behavior: ____________________

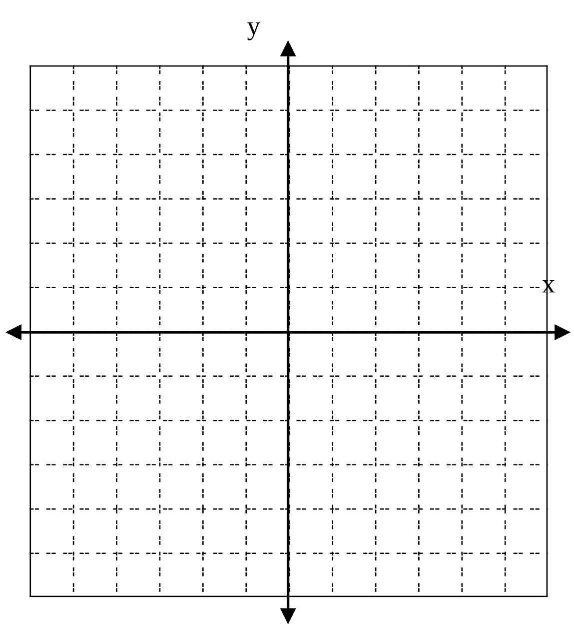
E: $f'(x) =$ ____________________

Critical Point(s): _____________________

F: $f''(x) =$ ____________________

Inflection Point(s): ___________________

Label Axis



Find each of the following, then sketch the graph. Show work/logic used.

NO GRAPHING CALCULATORS

1. $f(x) = \dfrac{x^2 - 2x + 4}{x - 2}$

A: Intercepts:

B: Symmetry:

x-intercept(s) ______________________

y-axis: Y/N _______________

y-intercept(s) ______________________

origin: Y/N _______________

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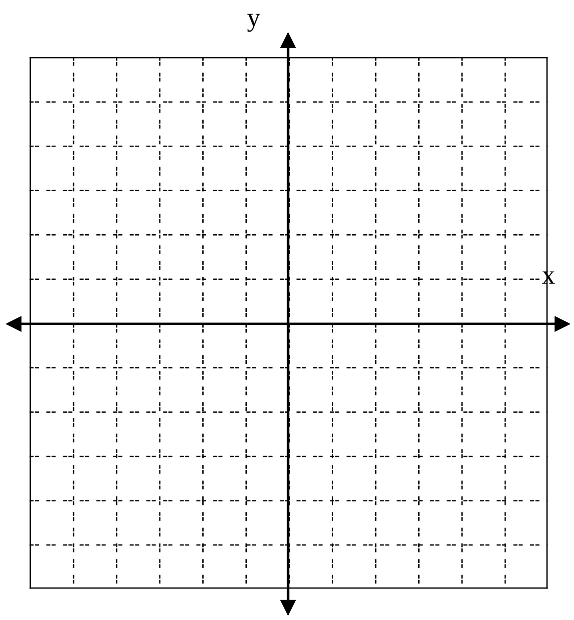
Inflection Point(s): _____________________

Label Axis



Find each of the following, then sketch the graph. Show work/logic used.

NO GRAPHING CALCULATORS

1. $f(x) = \dfrac{2(x^2 - 9)}{x^2 - 4}$

A: Intercepts:

x-intercept(s) ______________________

y-intercept(s) ______________________

B: Symmetry:

y-axis: Y/N _______________

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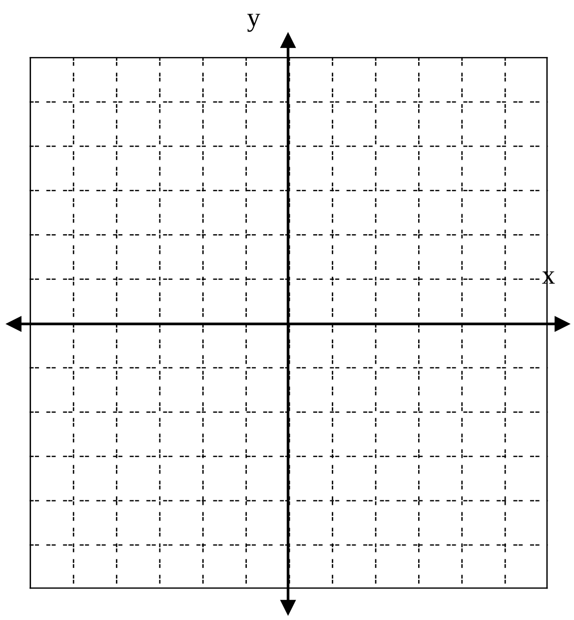

Find each of the following, then sketch the graph. Show work/logic used.

NO GRAPHING CALCULATORS

1. $f(x) = x^4 - 12x^3 + 48x^2 - 64x$

A: Intercepts:

x-intercept(s) ____________________

y-intercept(s) ____________________

B: Symmetry:

y-axis: Y/N ______________

origin: Y/N ______________

C: Domain ____________________

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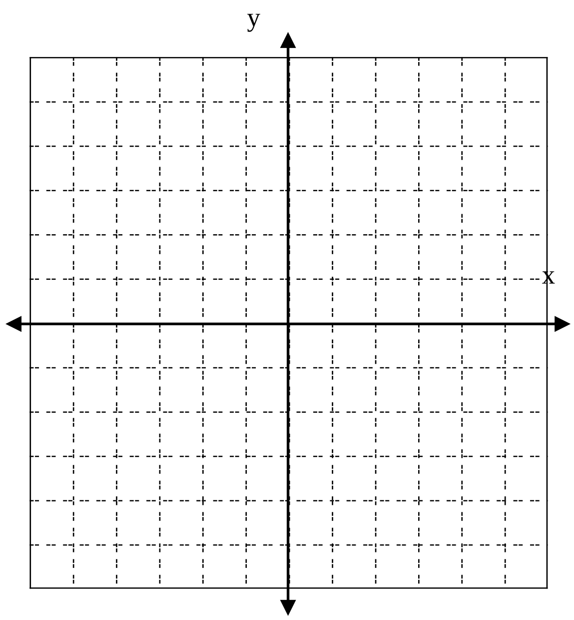
Label Axis



Find each of the following, then sketch the graph. Show work/logic used.

NO GRAPHING CALCULATORS

1. $f(x) = \dfrac{x}{\sqrt{x^2 + 2}}$

A: Intercepts:

x-intercept(s) ______________________

y-intercept(s) ______________________

B: Symmetry:

y-axis: Y/N _______________

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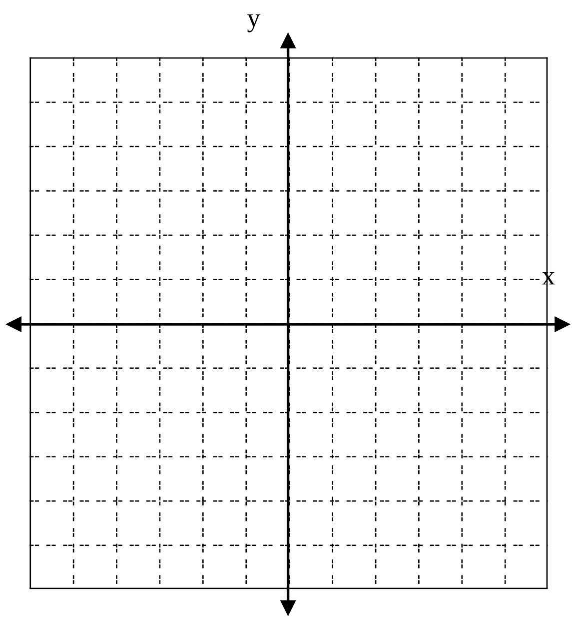
Label Axis



Find each of the following, then sketch the graph. Show work/logic used.

NO GRAPHING CALCULATORS

1. $f(x) = \dfrac{\cos(x)}{1 + \sin(x)} \quad \left(-\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$

A: Intercepts:

x-intercept(s) ______________________

y-intercept(s) ______________________

B: Symmetry:

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Inflection Point(s): _____________________

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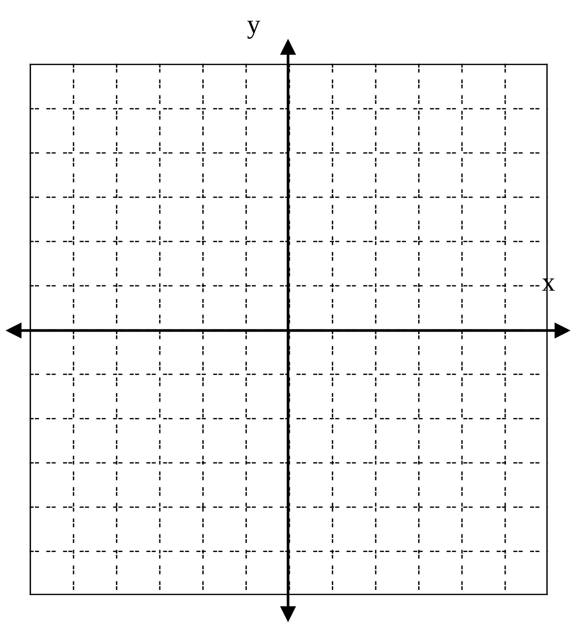

Find each of the following, then sketch the graph. Show work/logic used.

NO GRAPHING CALCULATORS

1. $f(x) = 2x^{\frac{5}{3}} - 5x^{\frac{4}{3}}$

A: Intercepts:

x-intercept(s) ______________________

y-intercept(s) ______________________

B: Symmetry:

y-axis: Y/N _______________

origin: Y/N _______________

C: Domain ______________________

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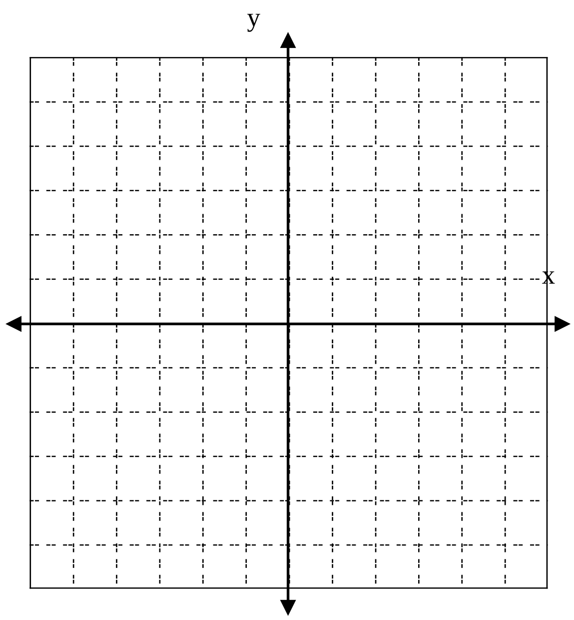
Inflection Point(s): _____________________

Label Axis



Find each of the following, then sketch the graph. Show work/logic used.

NO GRAPHING CALCULATORS

1. $f(x) = \dfrac{x^2}{\sqrt{x+1}}$

A: Intercepts:

x-intercept(s) ______________________

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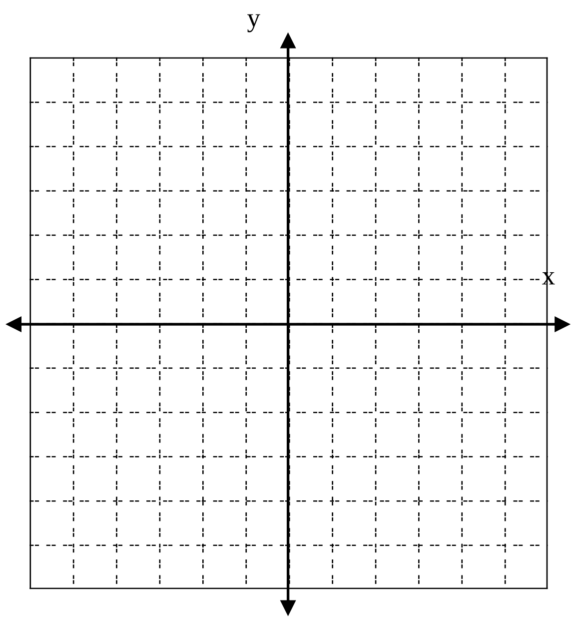

Find each of the following, then sketch the graph. Show work/logic used.

NO GRAPHING CALCULATORS

1. $f(x) = 2\cos(x) + \sin(2x)$ $[0,\ 2\pi]$

A: Intercepts:

x-intercept(s) ____________________

y-intercept(s) ____________________

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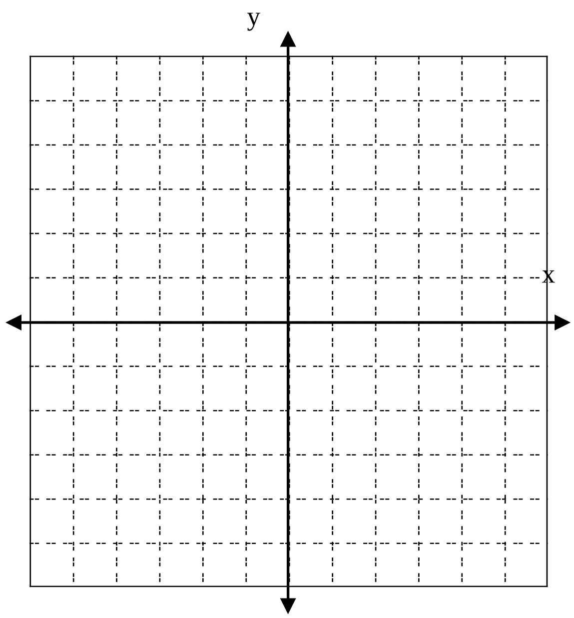

Find each of the following, then sketch the graph. Show work/logic used.

NO GRAPHING CALCULATORS

1. $f(x) = 3\sin(x) - \sin^3(x)$ $[-2\pi,\ 2\pi]$

A: Intercepts:

x-intercept(s) ____________________

y-intercept(s) ____________________

B: Symmetry:

y-axis: Y/N ______________

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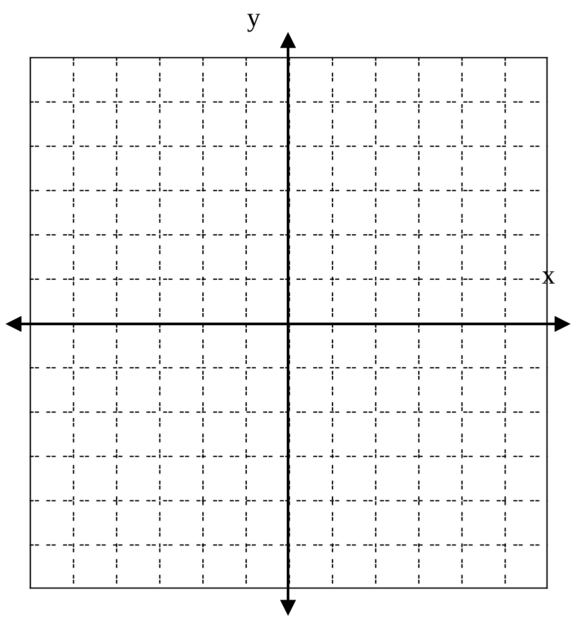

Find each of the following, then sketch the graph. Show work/logic used.

NO GRAPHING CALCULATORS **This One is Hard!**

1. $f(x) = x\tan(x)\ \left(-\frac{3\pi}{2}, \frac{3\pi}{2}\right)$

A: Intercepts:

x-intercept(s) ______________________

y-intercept(s) ______________________

B: Symmetry:

y-axis: Y/N _______________

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Critical Point(s): _______________________

F: $f''(x) =$ ______________________

Inflection Point(s): _____________________

Label Axis