Unit 2 Polynomials Day 1

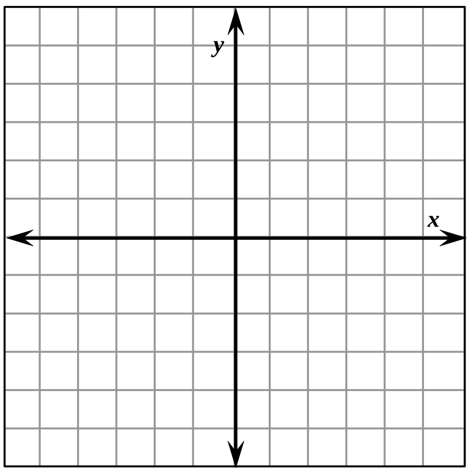
Quadratics: $f(x) = ax^2 + bx + c$

Vertex Form of Quadratic

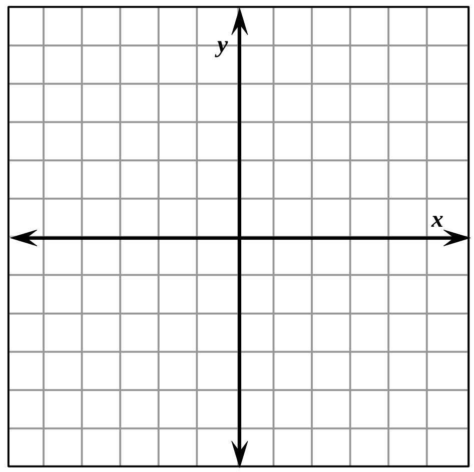
$f(x) = a(x - h)^2 + k$
Vertex (h, k)
Axis of Symmetry x = h (line)

Graphing:

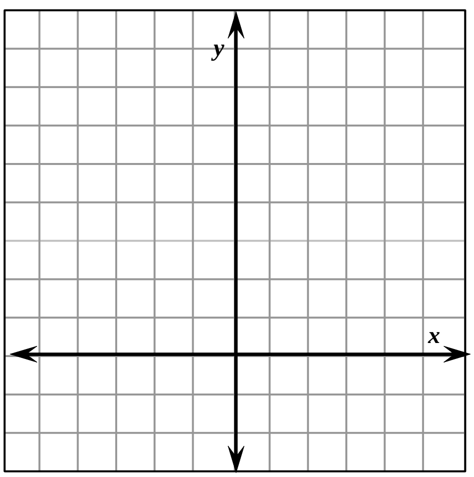
$f(x) = (x - 3)^2 + 2$



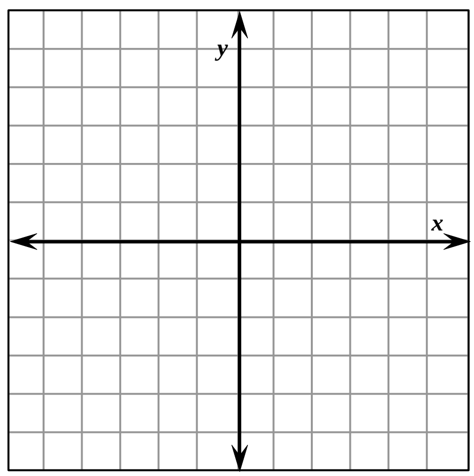
$g(x) = 2(x + 1)^2 - 4$



$h(x) = x^2 + 4$



$f(x) = 4 - (x - 1)^2$



Put each in vertex form by completing the square then find the intercepts.

Ex: $x^2 + 10x + 9 = 0$

Ex: $4x^2 - 16x - 2 = 0$

Ex: $-2x^2 - 12x + 5 = 0$

Ex: $3x^2 - 9x + 10 = 0$

Ex: $\frac{1}{2}x^2 + 3x - 10 = 0$

Ex: $\frac{3}{4}x^2 - 15x + 7 = 0$

Find the quadratic function that has the indicated vertex and passes through the given point.

a. Vertex: (4, -1); Point: (2,3)

b. Vertex: (2,3); Point: (5,-2)

1. Find two positive real numbers whose product is a maximum when the sum of the first and 3 times the second is 42.

2. Terry has 200 yards of fencing to enclose a rectangular garden on three sides. What dimensions of the garden will maximize the area?

3. A football is thrown by a quarterback to a receiver 40 yards away. The quadratic function *s(t)* = $-0.025t^2 + t + 5$ models the football's height above the ground, *s(t)*, in feet, when it is t yards from the quarterback. How many yards from the quarterback does the football reach its greatest height. What is that height?

Polynomial Functions:

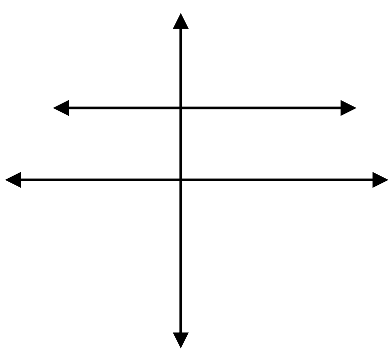
A polynomial in one variable, x, is an expression of the form $a_0 x^n + a_1 x^{n-1} + \ldots a_{n-2} x^2 + a_{n-1} x + a_n$ coefficients $a_0, a_1 \ldots$ represent complex numbers. a_0 is not zero and n is an nonnegative integer degree is n.

The graph of a polynomial function is **continuous**. This means that the graph of a polynomial function has not breaks, holes, or gaps. It only has smooth, rounded turns. It cannot have a sharp pointed turn.

Constant function

$y = 2$

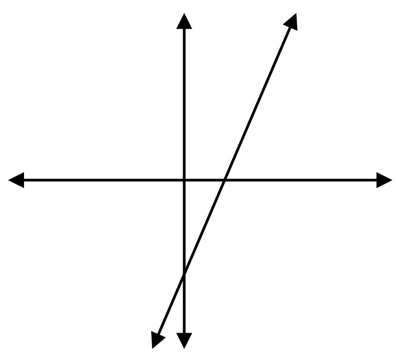
Degree 0



Linear function

$y = 3x – 5$

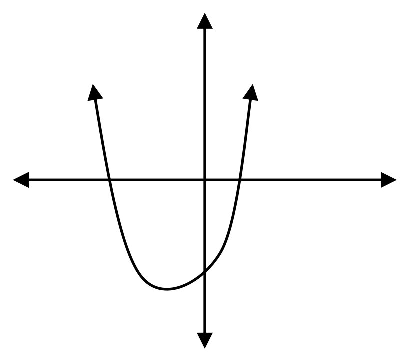
Degree 1



Quadratic function

$y = x^2 + 2x – 3$

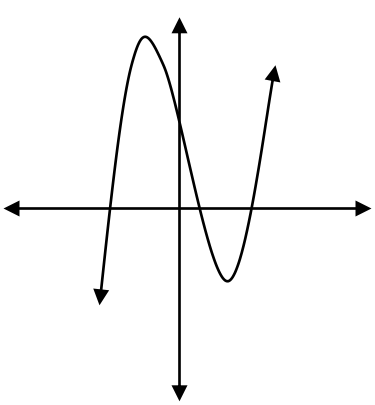
Degree 2



Cubic function

$y = x^3 – 5x + 2$

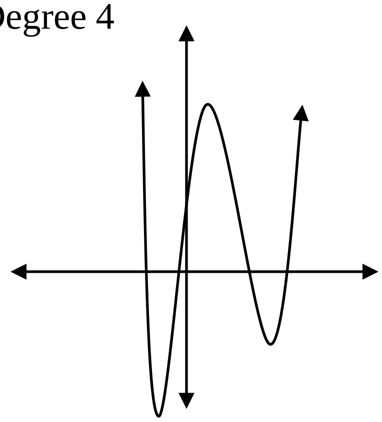
Degree 3



Quartic function

$y = x^4 – 3x^3 – 2x^2 + 7x + 1$

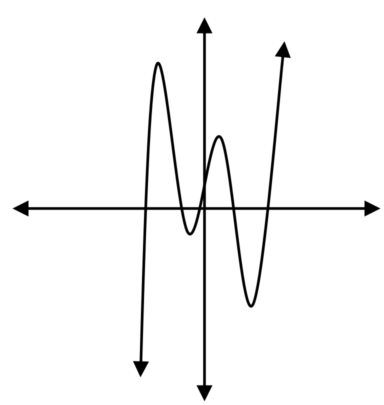
Degree 4



Quintic function

$y = x^5 – 5x^3 + 4x$

Degree 5



All have positive leading coefficient:

End Behavior Odd Degree: $\swarrow \nearrow$ Odd degree - starts low ends high

End Behavior Even Degree: $\nwarrow \nearrow$ Even degree - starts high ends high

If leading coefficient is negative:

End behavior Odd Degree: $\nwarrow \searrow$ Odd degree - starts high ends low

End behavior Even Degree: $\swarrow \searrow$ Even degree - starts low ends low

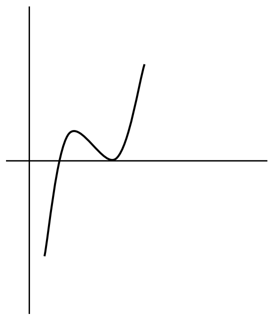
End Behavior Rules:

Leading Coefficient	$x \to -\infty$	$x \to \infty$
+, even power	$y \to \infty$	$y \to \infty$
-, even power	$y \to -\infty$	$y \to -\infty$
+, odd power	$y \to -\infty$	$y \to \infty$
-, odd power	$y \to \infty$	$y \to -\infty$

Zeros of Polynomial Functions:

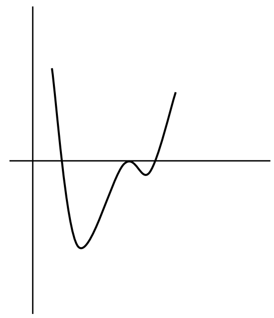
If f is a polynomial function, then the values of x for which f(x) is equal to 0 are called the zeros of f. These values of x are the roots, or solutions, of the polynomial equation $f(x) = 0$. Each real root of the polynomial equation appears as an x-intercept of the graph of the polynomial function.

1. If a polynomial $P(x)$ has a squared factor such as $(x - c)^2$, then $x = c$ is a **double root** of $P(x) = 0$ and the graph of $y = P(x)$ is tangent to the x-axis at $x = c$, as illustrated below.



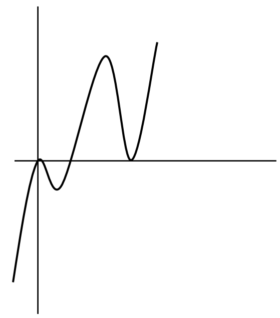
Cubic

$y = (x - 1)(x - 3)^2$



Quartic

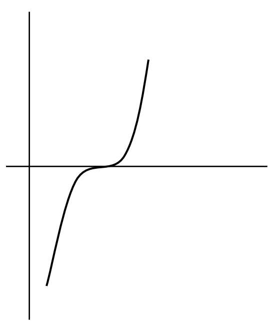
$y = (x - 1)(x - 3)^2(x - 4)$



Quintic

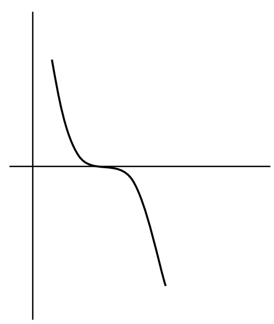
$y = x^2(x - 1)(x - 3)^2$

2. If a polynomial $P(x)$has a cubed factor such as$(x - c)^3$, then $x = c$ is a **triple root** of $P(x) = 0$ and the graph of $y = P(x)$ flattens out around $(c,0)$and crosses the x-axis at this point.



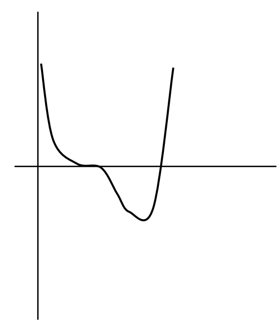
Cubic

$y = (x - 2)^3$



Cubic

$y = -(x - 2)^3$



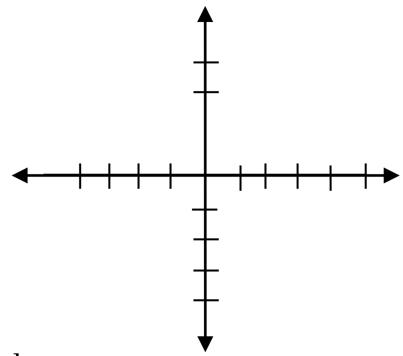
Quartic

$y = (x - 1)^3(x - 3)$

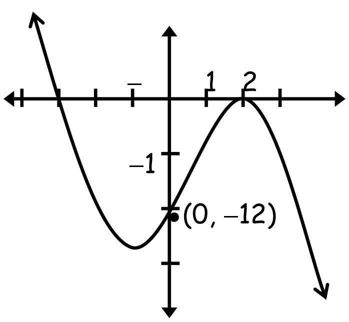
3. To quickly sketch the graph of a factored polynomial by hand:
   a. Draw a horizontal number line (x-axis). Plot the zeros (critical values) of the function on the number line. This separates the number line (x-axis) into intervals.
   b. Perform a sign analysis of $f(x)$ by testing one value of x from each of the intervals determined by the zeros. If the sign of a tested value is positive, then the graph lies above the x-axis in that interval. If the sign is negative, then the graph lies below the x-axis in that interval.
   c. Sketch the graph. If no particular point (other than the zeros) is given, don't be concerned with exactly how high/low the graph rises/falls above/below the x-axis.

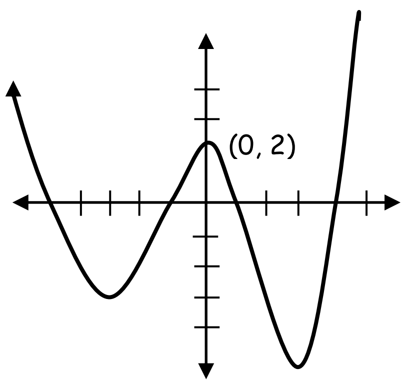
Day 2

4. Factor $f(x) = x^3 - 2x^2 - 4x + 8$ and sketch its graph.



5. Write the equation for the polynomial graph shown below.





6. Find a polynomial function that has the given zeros.

a. 0,2,7 b. 4,-3,3 c. -2, -1, 3, 2

Remainder Theorem — for $f(x)$, the value of $f(a)$ is equal to the remainder when $f(x)$ is divided by $x - a$.

Ex: $f(x) = 3x^4 + x^3 - 2x^2 - 5$

Ex: If $f(x) = 3x^5 + 6x^4 - 75x + 10$, $f(-3)$ using synthetic sub.

$f(2) =$

What is the remainder when $P(x) = x^{15} + 3x^{10} + 2$ is divided by $x - 1$.

Factor Theorem: the binomial x – a is a factor of the polynomial

$$f(x) \Leftrightarrow f(a) = 0$$

Ex: Is x – 2 a factor of $f(x) = x^4 - 4x^3 + 5x^2 + 4x - 12$

Find $f(2)$, if it equals zero, then x – 2 is a factor!

Ex: Factor $f(x) = 3x^3 + 14x^2 - 28x - 24$ given that x – 2 is a factor.

Ex: $x = -4$ $x = -1$ $x = 2$

$f(x) = (x + 4)(x + 1)(x - 2)$

Ex: If x + 2 is a factor of $x^3 - x^2 - 10x - 8$. Find the others.

Roots and Zeros:

Ex: Given that $-\frac{3}{2}$ is a root, solve $12x^3 + 6x^2 - 20x - 3 = 0$

Ex: $w^3 + w^2 - 6w - 120$

The real zeros will be the x-intercepts of the graph.

The roots will be 5, $-3 + i\sqrt{15}$, $-3 - i\sqrt{15}$

One real and two imaginary

Remember they belong to the set of Complex #'s.

Ex: Given that -2 is a zero of $f(x) = x^3 - 2x^2 + 5x + 26$, find all other zeros.

Ex: Solve $p(x) = x^3 + x + 10$, if -2 is a root.

Fundamental Theorem of Algebra: Every polynomial equation with degree greater than zero has at least one root in the set of complex numbers(includes real numbers).

Therefore, a polynomial equation of the form $P(x) = 0$ of degree n, has exactly n roots.

# of solutions = degree of the polynomial

☆ **Every polynomial of ODD degree (with real coefficients) has at least one REAL root** ☆

Ex. $x^3 + 4x^2 + 4x = 0$ has 3 solutions

Ex. $x^4 - 10x^2 + 9 = 0$ has 4 solutions

***imaginary roots occur in CONJUGATE PAIRS (if polynomial coefficients are real)**

The multiplication of complex conjugates gives a real number.

$$(a + bi)(a - bi) = a^2 + b^2$$

If 2 + 3i is a root, then so is 2 – 3i

**Multiplicity: how many times a solution repeats.

Ex. $x^2(x - 4)(x - 4)(x - 4) = 0$

Solutions: {0(mult. of 2), 4(mult. of 3)}

Day 3

Definition of a Complex Number:
The set of all numbers in the form $a + bi$ with real numbers a and b, and i, the imaginary unit. The real number a is called the real part, and the real number b is called the imaginary part. If $b \neq 0$, then complex number is called an imaginary number. An imaginary number is the form of bi is called a pure imaginary number.

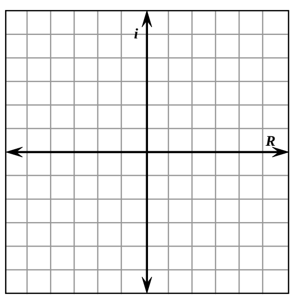
Divide.

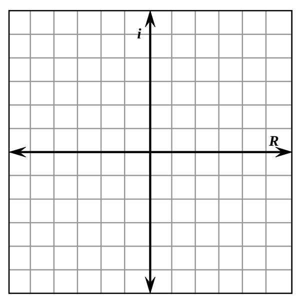
a. $\frac{7+4i}{2-5i}$ b. $\frac{5+4i}{4-2i}$

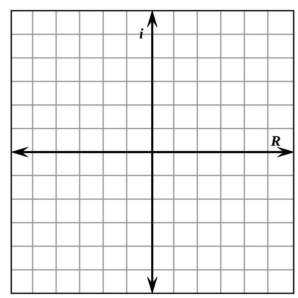
Complex Plane

Plot the complex number in the complex plane.

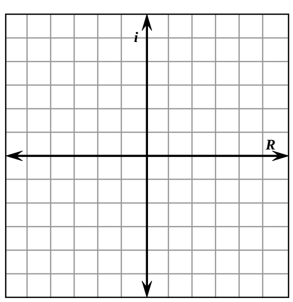
a. $2 + 3i$ b. $-1 + 4i$ c. $-4 - i$

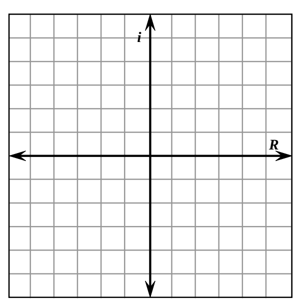


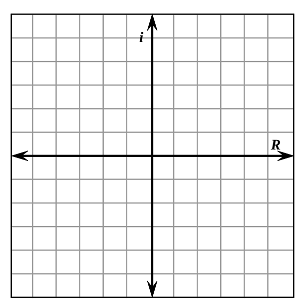




d. $5 - 2i$ e. $5i$ f. -2







Ex: Find all zeros of $f(x) = x^3 - 11x^2 + 40x - 50$, if $3 + i$ is one zero of $f(x)$.

Ex: Solve $x^4 + x^3 + 6x^2 – 14x – 20 = 0$ if $–1 + 3i$ is a root.

Writing an equation given the zeros:

Ex: Find a <u>cubic equation</u> with integral coefficients that has 2 and $3 – i$ as roots.

Ex: Given {4, -1, 3i} write an <u>equation</u> with integral coefficients.

Descartes' Rule Of Signs:

If $p(x)$ is a polynomial with real coefficients, then

a) the number of positive real roots is either equal to the number of variations of sign of $p(x)$ or is less than this number by a positive even integer.

Example: $p(x) = x^4 + 8x^3 – 2x^2 – 3x + 8$

There are 2 variations in sign $\therefore$ there are 2 or 0 positive real roots.

b) the number of negative real roots is either equal to the number of variations of sign of $p(-x)$ or less than this number by a positive even integer.

Example: $p(-x) = (-x)^4 + 8(-x)^3 – 2(-x)^2 – 3(-x) + 8 = x^4 – 8x^3 – 2x^2 + 3x + 8$

There are 2 variations in sign $\therefore$ there are 2 or 0 negative real roots.

Possibilities for types of roots:

+	-	imag.
2	2	0
2	0	2
0	2	2
0	0	4

All rows must add up to 4

(degree of polynomial)

Day 4

Descartes Rules of Signs

Ex: $g(x) = 3x^4 - x^3 + 8x^2 + x - 7$

Ex: $f(x) = x^3 + 2x^2 - 5x - 6$

Ex: $f(x) = x^5 - x^3 - x + 1$

Ex: $f(h) = h^3 + 7h^2 - 36$

1 positive root

Rational Zero Theorem-

$f(x) = a_0 x^n + a_1 x^{n-1} + \ldots a_{n-1} x + a_n$ represents a polynomial function

If $\frac{p}{q}$ is a rational number in simplest form and is a zero of $y = f(x)$,

then p is a factor of a_n and q is a factor of a_0.

List all possible rational zeros of $f(x) = 2x^3 - x^2 - 34x - 56$

Upper & Lower Bounds:

Real roots of $p(x)$ lie in the interval between least positive upper bound and greatest negative lower bound.

If $p(x)$ has real coefficients and a positive leading coefficient to determine the upper and lower bounds of the polynomial equation:

*when $p(x)$ is divided by $x - b$ using synthetic division, where $b > 0$, if the row that contains the quotient and remainder are nonnegative(could be zero), then b is an upper bound for the real zeros of $p(x)$.

**when $p(x)$ is divided by $x - a$, where $a < 0$, if the row that contains the quotient and the remainder alternate in sign(non-positive and non-negative), then a is a lower bound for the real zeros of $p(x)$.

(Alternately greater than or equal to zero and less than or equal to zero)

Determine the least positive integral upper bound and the greatest negative integral lower bound of: $3x^3 - 4x^2 - 4x + 4 = 0$

Testing for upper bound:

Start dividing using positive numbers (1, 2, 3, …):

	3	-4	-4	4	
1	3	-1	-5	-1	
2	3	2	0	4	← all values are ≥ 0

$\therefore$ b = 2 (least positive upper bound)

Testing for lower bound: (divide using –1, -2, -3, …)

	3	-4	-4	4
-1	3	-7	3	1
-2	3	-10	16	-28

$\longleftarrow$ alternately ≥ 0 and ≤ 0

$\therefore$ a = -2 (greatest negative lower bound)

Real roots lie in the interval <u>between</u> -2 and 2.

Ex: $2x^4 - 6x^2 - 3x - 9 = 0$

Upper

	2	0	-6	-3	-9
1					
2					
3					

Lower

	2	0	-6	-3	-9
-1					
-2					
-3					

Ex: $x^4 - 3x^3 - x^2 + 3x - 1 = 0$

Upper	1	-3	-1	3	-1
1					
2					
3					
4					

Lower	1	-3	-1	3	-1
-1					
-2					
-3					
-4					

To test for lower bound a row of 1 -4 3 0 -1 would not be alternating but

1 -4 3 0 1 would be alternating ($\geq 0, \leq 0, \geq 0, \leq 0, \geq 0$)

Ex: $f(x) = x^3 + 4x^2 - 25x - 28$

p is a factor -28 $\pm 1, \pm 2, \pm 4, \pm 7, \pm 14, \pm 28$

q is a factor 1 ± 1

Now try your choices by synthetic division.

R |	**1**	**4**	**-25**	**-28**
1	1	5	-20	**-48**
2	1	6	-13	**-54**
-2	1	2	-29	**30**
-1	1	3	-28	

So, -1 is a zero x + 1 is a factor

Now $x^2 + 3x - 28 = 0$

$(x + 7)(x - 4) = 0$

$x = -7 \quad x = 4$

So (x + 7) (x – 4) (x + 1) are factors

Ex: Find all zeros of $f(x) = 2x^4 - 9x^3 + 2x^2 + 21x - 10$

Day 5 **All You Need to Know About Rational Functions**

The equation of two polynomials, $\frac{g(x)}{f(x)}$ where $f(x) \neq 0$ Parent Rational Function is $\frac{1}{x}$.

Definition of a Vertical and Horizontal Asymptotes

a. The line $x = a$ is a vertical asymptote of the graph of f if $f(x) \to \infty$ *or* $f(x) \to -\infty$ as $x \to a$, either right or from the left.

b. The line $y = b$ is a horizontal asymptote of the graph of f if $f(x) \to b$ *as* $x \to -\infty$ *or* $x \to -\infty$.

1. **Holes:** Occur when there is a common FACTOR in the numerator and denominator. To find the exact location of the hole, set the common factor equal to zero, then take that value and substitute it back into the **SIMPLIFIED** rational equation.

Ex: $h(x) = \frac{5(x-2)}{(x+4)(x-2)}$ therefore the hole is located at (2, *f*(2)) which is $\left(2, \frac{5}{6}\right)$.

1. **Vertical Asymptotes:** these occur at the x-value that makes the denominator equal to zero.

Ex: $g(x) = \frac{5}{x-2}$ has a vertical asymptote at x = 2.

Ex: $f(x) = \frac{3x}{(x-4)(x+2)}$ has two vertical asymptotes one at x = 4 and one at x = -2.

2. **Horizontal Asymptotes:** A rational function has **at most** one horizontal asymptote.

Three conditions: Given $f(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1} .. a_0}{b_m x^m + b_{m-1} x^{m-1} ... b_0}$

1. if n > m then f has no horizontal asymptote (**it may have a slant asymptote**)

Ex. $p(x) = \frac{3x^2 + 4}{x+2}$

2. if n = m then f has a horizontal asymptote at y = $\frac{a^n}{b^m}$ (leading coefficients)

Ex: $g(x) = \frac{3x^2 + 4}{4x^2 + 2x - 5}$ horizontal asymptote at y = $\frac{3}{4}$.

3. if n < m then y = 0 is the horizontal asymptote

Ex: $f(x) = \dfrac{2x+1}{3x^2+6x+2}$ horizontal asymptote at y = 0.

Slant Asymptote: They are exactly how they sound. *Neither vertical or horizontal!! These occur only if the degree of the numerator is <u>exactly</u> one more than the degree of the denominator.

Need to do synthetic or long division to find it.

Ex. $p(x) = \dfrac{x^2-2x+1}{x+3}$

Ex: $w(x) = \dfrac{3x^2+4}{x+2}$

3. Determine the vertical and horizontal asymptotes of the function, (include slant, if one). Also list any holes, if any. Then state the domain.

a. $f(x) = \dfrac{5x}{x-4}$

Hole:
VA:
Domain:
HA:
SA:
x-intercept:
y-intercept:

b. $f(x) = \dfrac{-2x+1}{3x+5}$

Hole:
VA:
Domain:
HA:
SA:
x-intercept:
y-intercept:

c. $f(x) = \dfrac{3}{x^2+4}$

Hole:
VA:
Domain:
HA:
SA:
x-intercept:
y-intercept:

d. $g(x) = \dfrac{x^2-1}{x^2-6x-7}$

Hole:
VA:
Domain:
HA:
SA:
x-intercept:
y-intercept:

e. $g(x) = \dfrac{x^2-6x+9}{x^2-x-6}$

Hole:
VA:
Domain:
HA:
SA:
x-intercept:
y-intercept:

f. $f(x) = \dfrac{x^2+6x+8}{x+4}$

Hole:
VA:
Domain:
HA:
SA:
x-intercept:
y-intercept:

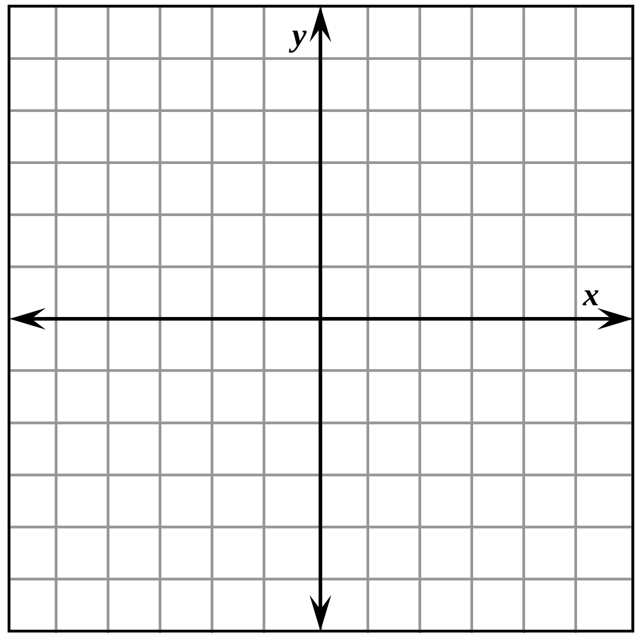
4. Write an equation of a function that has a vertical asymptote at $x = -1$ and a hole at $x = 4$

5. When graphing a rational function that is not a standard transformation of $\frac{1}{x}$ then there are some steps to take to make it easier to graph, without using a graphing calculator.

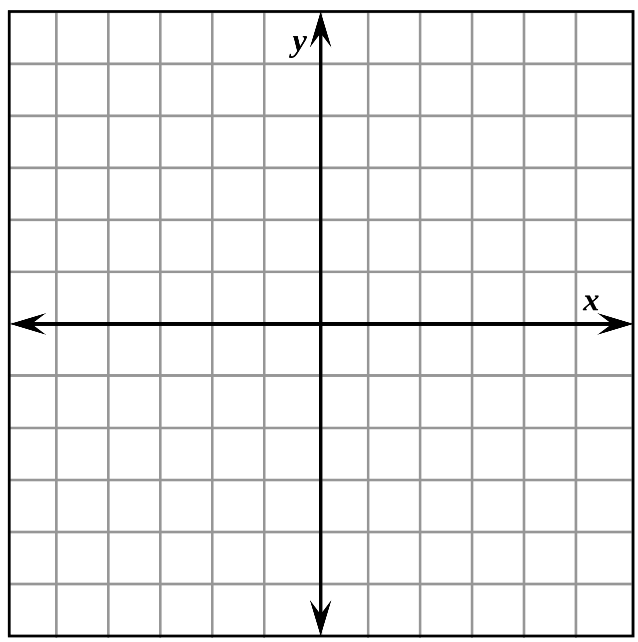
   a. Factor and simplify if possible. Look for any Holes!
   b. Identify any Vertical, Horizontal, Slant asymptotes.
   c. Find the y-intercept, if one
   d. Find the x-intercepts, if any.
   e. Start to sketch the function with the information you have!

6. Graph each function.

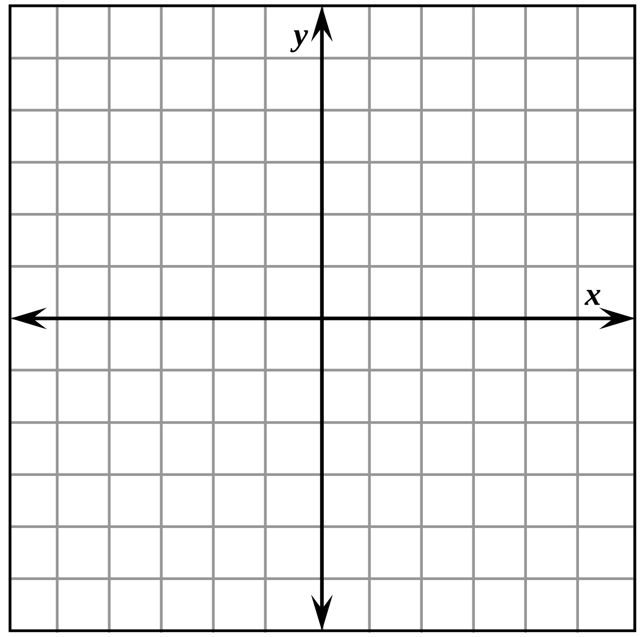
   a. $f(x) = \dfrac{3x-2}{x+3}$



   b. $f(x) = \dfrac{2x^2-5x-7}{x-2}$



c. $f(x) = \dfrac{2x^3}{x^2+1}$



Day 6

Write the equation of the rational function having these characteristics.

a) vertical asymptotes at x = 4
b) x intercepts at (1, 0)
c) horizontal asymptote at y = 3/5

Write the equation of the rational function having these characteristics.

a) vertical asymptotes at x = 4 and x = -1
b) x intercepts at (3, 0), (-2, 0)
c) horizontal asymptote at y = 2/3
d) y intercept at (0, 1)

1. Find the slant asymptote of each.

a. $f(x) = \dfrac{x^2+3x+1}{x}$ b. $p(x) = \dfrac{x^2-2x+1}{x+4}$ c. $m(x) = \dfrac{x^2-4x-5}{x-3}$

What happens here? $f(x) = \dfrac{x^4+3x^3+5x^2-7x+1}{x^2-x+5}$

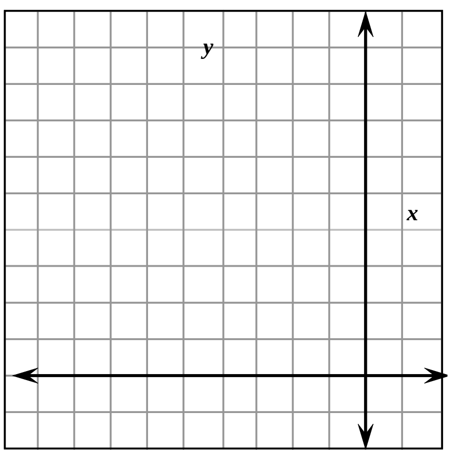
2. Write the equation of the rational function having these characteristics.

a) vertical asymptotes at x = 0
b) x intercepts at (3/2, 0), (-4, 0)
c) slant asymptote at y = 2x + 1 (Tricky!)

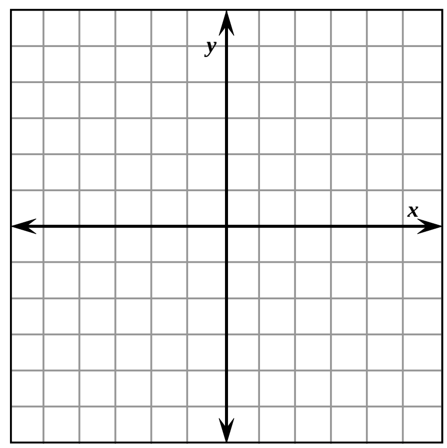
For each function below, find the following:

a. x and y intercepts
b. vertical asymptotes
c. slant asymptotes, if one
d. Holes
e. List the Domain and Range (in interval notation)
f. Sketch a complete graph by showing test points in various regions of the graph.

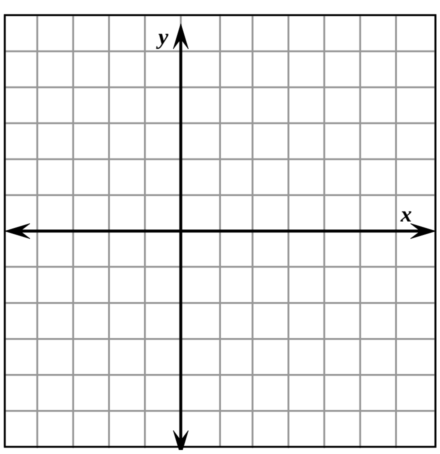
2. $f(\mathrm{x}) = \dfrac{3x-2}{x+3}$



3. $h(\mathrm{x}) = \dfrac{-2x^2+3x+2}{x^2-x-12}$



4. $h(x) = \dfrac{x^3-1}{x^2-9}$



5. $f(x) = \dfrac{2x^2-5x-7}{x-2}$