Differential Equations – Separation of Variables

If y is a differentiable function of t such that y > 0 and $y' = ky$, for some constant k, then $y = Ce^{kt}$.

C is the initial value of y, and k is the proportionality constant. Exponential growth occurs when k > 0, and decay occurs when k < 0.

Ex: Carbon (14) has a half-life of 5,730 years. If the initial value of y is 32,000. How much of the substance will be left after 100 years, 42,000 years?

$y = Ce^{kt}$ $\quad k = \ln(1/2)/5{,}730$

$y = 3200\,e^{(\ln(1/2)/5{,}730)\cdot t}$

100 years = 3,161.5 $\quad$ 42,000 years = 19.9

Ex: The rate of change of *N(t)* number of bears in a population is proportional to $1200 - N(t)$, $t \geq 0$ is time in years. $N(4) = 600$, $N(0) = 300$

$\int \frac{dN}{1200 - N} = \int k\,dt$

$-\ln|1200 - N| = kt + C$

A) Write the differential equation. $\frac{dN}{dt} = k(1200 - N)$

B) Solve the equation. $N(t) = 1200 - 900e^{-kt}$

$N(t) = 1200 - Ce^{-kt}$

$\boxed{k = \tfrac{1}{4}\ln(3/2)}$

$\boxed{\begin{array}{c} C = 900 \\ N(0) = 300 \end{array}}$

C) What is *N(8)*? $N(8) \approx 800$

$N(t) = 900\,e^{-\frac{1}{4}\ln(3/2)t}$

D) Find the limit as t goes to infinity of *N(t)*. $\lim_{t \to \infty} N(t) = 1200$

Ex: Suppose an experimental population of flies increases according to the exponential of growth. There were 100 flies after the second day of the experiment and 300 flies after the fourth day. Approximately how many flies were in the original population?

$y = Ce^{kt}$ $\quad t = 2 \quad y = 100$

$C = 100e^{-2k} \Leftarrow 100 = Ce^{2k}$ $\quad t = 4 \quad y = 300$

$\rightarrow 300 = Ce^{4k}$

$300 = 100e^{2k}$ $\quad k = \tfrac{1}{2}\ln 3 \approx .5493$

$y = Ce^{.5493t}$ $\quad (2, 100)$ $\quad C \approx 33$

Ex: A certain type of bacteria increases continuously at a rate proportional to the number present at time t. If there are 500 at a given time and 1000 2 hours later, how many hours will it take for there to be 2500?

$P = P_0 e^{kt}$ $\quad P = 500\,e^{\frac{\ln 2}{2}\cdot t}$

$P = 500e^{kt}$ $\quad 2500 = 500\,e^{\frac{\ln 2}{2}\cdot t}$

$1000 = 500e^{2k}$ $\quad 5 = e^{\frac{\ln 2}{2}\cdot t}$

$2 = e^{2k}$ $\quad \ln 5 = \frac{\ln 2}{2}\cdot t$

$\frac{\ln 2}{2} = k$ $\quad t = \frac{2\ln 5}{\ln 2}$

Ex: Newton's Law of Cooling states that the rate of change in the temperature of an object is proportional to the difference between the object's temperature and the temperature of the surrounding medium. Let y represent the temperature of an object in a room whose temperature is kept at a constant 70 degrees. If the objects cools from 120 degrees to 100 degrees in 15 minutes, how much longer will it take for its temperature to decrease to 80 degrees?

$$y' = k(y-70)$$

$$\frac{dy}{dt} = k(y-70) \Rightarrow \int \frac{dy}{y-70} = \int k\,dt$$

$$\ln|y-70| = kt + C$$

$$y - 70 = Ce^{kt}$$

$t=0 \quad y=120$

$$y = Ce^{kt} + 70$$

$$120 = C + 70$$

$$50 = C$$

$$100 = 50e^{15k} + 70$$

$$3/5 = e^{15k}$$

$$1/15 \ln 3/5 = k$$

$$-.034 = k$$

$$80 = 50e^{-.034t} + 70$$

$t = 47.3$ mins.

Ex: An orthogonal trajectory of a family of curves is a curve that intersects each curve of the family orthogonally, at right angles. Find the orthogonal trajectories of the family of curves $x = ky^2$, where k is an arbitrary constant.

$$x = ky^2$$

$$1 = 2ky\frac{dy}{dx}$$

$$\frac{1}{2ky} = \frac{dy}{dx}$$

$$\frac{1}{2\left(\frac{x}{y^2}\right)y} = \frac{dy}{dx}$$

$$\frac{y}{2x} = \frac{dy}{dx}$$

orthogonal $\Rightarrow \frac{-2x}{y} = \frac{dy}{dx}$

$$\int y\,dy = \int -2x\,dx$$

$$\frac{1}{2}y^2 = -x^2 + C$$

$$x^2 + 1/2\,y^2 = C$$

(Family of ellipses)

Ex: Electro-static fields and streamlines