Precalculus Sequences & Series

A **sequence** is an ordered list of numbers. Each number in the list is called a **term** of the sequence. The first term of a sequence is denoted as a_1. The second term is denoted as a_2. The term in the nth position is called the nth term and is denoted as a_n. The term before a_n is a_{n-1}.

A sequence is a function whose range is the terms of the sequence and the domain is the position of each term.

There are many types of sequences:

1) An **infinite sequence** is a sequence with an infinite number of terms.
Can we come up with some examples?

Ex: 2, 4, 6, 8, 10, 12,....

Ex: 1, -3, 5, -7, 9, -11, 13 .....

2) A **finite sequence** is a sequence with a finite number of terms. The sequence ends at a certain point.

Three sequences we will learn about:

3) An **Arithmetic sequence** is a sequence in which the difference between each term and the proceeding term is always constant. This difference is known as *d* and is constant term.

4) A **Geometric sequence** is a sequence in which terms are found by multiplying a preceding term by a non-zero constant. This term is known as *r* and is called the common ratio of a geometric sequence.

5) A **Recursive Sequence** is a sequence in which each term is defined using the previous terms.

I. Write a rule for the nth term for the following examples:

6) Ex 1. 1, 2, 3, 4, 5, ...

adding one

$a_n = 1n + 0$

Ex. 2 2,4,6,8,10,...

$a_n = 2n + 0$

Ex.3 2,3,4,5,6,...

$a_n = n + 1$

7) Ex. 4 5,8,11,14,...

$a_n = 3n + 2$

Ex. 5 1,3,5,7,9,...

$a_n = 2n - 1$

Ex. 6 $\frac{2}{5}, \frac{2}{25}, \frac{2}{125}, \frac{2}{625}...$

multiplication

$a_n = 2/5 \left(\frac{1}{5}\right)^{n-1}$

or $a_n = 2(1/5)^n$

II. List the first four terms of the sequence given by

Ex: $a_n = 4n - 3$

$a_1 = 1$
$a_2 = 5$
$a_3 = 9$
$a_4 = 13$

Ex: $a_n = \frac{(-1)^n}{2n-1}$

$a_1 = \frac{-1}{1}$
$a_2 = \frac{1}{3}$
$a_3 = \frac{-1}{5}$
$a_4 = \frac{1}{7}$

Ex: $a_n = 7\left(3^{-n}\right)$

$a_1 = 7/3$
$a_2 = 7/9$
$a_3 = 7/27$
$a_4 = 7/81$

Arithmetic sequences:

The sequence $a_1, a_2, a_3, a_4, \ldots a_n$ is arithmetic if there is a number d such that:

$a_2 - a_1 = d$
$a_3 - a_2 = d$ Where d is the common difference.

Ex: 6, 9, 12, 15, ….3n + 3 The common difference is 3 because 9 – 6 = 3. d =3

Ex: 2, -3, -8, -13, …, 5n + 7 The common difference is –5 because -3–2 = -5. d = -5

Explicit Formula: a formula that defines the n^{th} term.

slope y-int

$a_n = dn + a_0$

← (term in front of seq)

The n^{th} term of an arithmetic sequence: $a_n = dn + c$

Is the form for the n^{th} term of an arithmetic sequence. Where d is the common difference and $c = a_1 - d$ (The first term minus the common difference).

Your book represents this differently ($a_n = a_1 + (n-1)d$ or $a_n = a_{n-1} + d$)

Ex: Find a formula of an arithmetic sequence whose common difference is 4 and whose first term is 3.

$a_n = dn + c$ We know d = 4. $a_1 = 3$. So c = 3 – 4. c = -1

$a_n = 4n - 1$. The terms of this sequence are: 3, 7, 11, 15, …, 4n – 1.

Ex: Find the formula of the arithmetic sequence whose first term is 3 and whose second term is –1.

$a_n = dn + c$ We know $a_1 = 3$ and $a_2 = -1$. So d = -4. c must be 3–(-4) = 7

$a_n = -4n + 7$

I. Write a rule for the n^{th} term for the following examples:

Ex: 2,4,6,8,10,…

$a_n = 2n + 0$

Ex: $\overset{\frac{4}{10}}{\frac{2}{5}}, \frac{9}{10}, \overset{\frac{14}{10}}{\frac{7}{5}}, \frac{19}{10} \ldots$

$d = 5/10$

$a_n = \frac{1}{2}n - \frac{1}{10}$

Ex: 4,8,12,16,…

$a_n = 4n + 0$

Ex 1: The fifth term of an arithmetic sequence is 25 and the 12^{th} term is 60. Write the first several terms of this sequence.

$a_5 = 25$ $a_{12} = 60$
$a_{12} = a_5 + 7d$ Where 7 is the difference in the term numbers.
$60 = 25 + 7d$
$35 = 7d$
$5 = d$

slope $\frac{60-25}{12-5} = \frac{35}{7} = 5$

Since $a_5 = 25$ we can subtract 5 to get each term in the sequence down to the first.

5, 10, 15, 20, 25

Ex 2: Find the eighth term of an arithmetic sequence that begins with 1 and 7.

$d = 7 - 1 = 6$

Method 1: Write out the first 8 terms. 1, 7, 13, 19, 25, 31, 37, 43

Method 2: Find the nth term by finding c.

$a_n = dn + c$ $\quad c = a_1 - d \quad c = 1 - 6$ or -5

$a_n = 6n - 5$

So $a_8 - 6(8) - 5 = 43$

Ex 3: Find a rule for the n^{th} term and fill in the missing terms.

-2, 4, ___, ___, 22, ___ $\quad a_n = 6n - 8$

$3d + 4 = 22 \quad d = 6$

Ex 4: $a_7 = 34$ $a_{18} = 122$ Write a rule for the nth term. $\quad d = \frac{122-34}{18-7} = \frac{88}{11} = 8$

$a_n = 8n - 22 \quad a_7 = 8(7) + a_0 \quad 34 = 56 + a_0 \quad a_0 = -22$

Arithmetic means: the terms between any two nonconsecutive terms of an arithmetic sequence.

The terms between 2 given terms of an arithmetic sequence are called *arithmetic means*.

$10, \underbrace{13, 16, 19}_{\text{3 arithmetic means}}, 22$

$10, \underbrace{14, 18}_{\text{2 arithmetic means}}, 22$

Ex 5. Insert 4 arithmetic means between 15 and 50.

15, 22, 29, 36, 43, 50

$5d + 15 = 50$

$5d = 35 \quad d = 7$

Ex 6: Form an arithmetic sequence that has six arithmetic means between -12 & 23.

-12 __ __ __ __ __ __ 23 $\quad 7d - 12 = 23$

$a_n = 5n - 17 \quad d = 5$

Summation Notation: the sum of a sequence is also known as an **Arithmetic Series**.

$$\sum_{k=1}^{m} c_k = c_1 + c_2 + c_3 + \ldots + c_m$$

Sigma Notation: the sum of the first n terms of a sequence (called a series)

Ex: $\displaystyle\sum_{i=2}^{5} 4i$ (5 ← stop, $i=2$ ← start)

$4(2) + 4(3) + 4(4) + 4(5)$

$8 + 12 + 16 + 20$

$= 56$

Ex: $\displaystyle\sum_{k=3}^{8} (4 - k)$

$k=3 \quad k=4 \quad k=5 \quad k=6 \quad k=7 \quad k=8$

$1 + 0 + -1 + -2 + -3 + -4$

$= -9$

Partial Sums of an **Arithmetic Sequence**:

$S_n = \frac{n}{2}(a_1 + a_n)$ This means that we add the first and last terms, then multiply by the number of terms divided by 2.

Ex: Find the sum of 2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 + 18 + 20.

We know that this is an arithmetic sequence because d = 2 and there are 10 terms.

$n = 10$, $a_1 = 2$ and $a_n = 20$

$S_n = \frac{10}{2}(2 + 20) = 5(22) = 110$

Gauss: asked to add number from 1 to 100 in 3rd grade.

1 + 2 + 3 + 4 + 5 + … + 98 + 99 + 100

101

101

101

(50 pairs)(101) = 5050 or ______________

Ex: Find the sum of the integers from 1 to 500.

$S_n = \frac{n}{2}(a_1 + a_n)$ $n = 500$, $a_1 = 1$ and $a_n = 500$

$S_n = \frac{500}{2}(1 + 500) = 250(501) = 125,250$

Ex: Find the sum of the first 200 terms of the arithmetic sequence: 3, 7, 11, 15…..

$S_n = \frac{n}{2}(a_1 + a_n)$ $n = 200$, $a_1 = 3$, $a_n = ?$, and $d = 4$

1) Find a_n

$a_n = dn + c$ Where $c = 3 - 4 = -1$

$a_n = 4n + -1$

2) Find the 200th term

$a_{200} = 4(200) - 1 = 800 - 1 = 799$

3) Find S_n

$S_n = \frac{200}{2}(3 + 799) = 100\,(802) = 80,200$

Ex: Find the sum of the arithmetic series where $a_1 = -111$, $d = 3$, and $a_n = 9$

$n = ?$ $S_n = ?$

$a_n = 3n - 114$

$9 = 3n - 114$ $n = 41$

$S_n = \frac{41}{2}(-111 + 9)$

$= -2091$

Ex: Find the first three terms of the arithmetic series where $a_1 = 10$, $a_n = -46$, and $S_n = -522$.

$n = ?$

$d =$

$-522 = \frac{n}{2}(10 + -46)$

$29 = n$

$a_{29} = 28(d) + 10$

10 8 4

Recursive Sequences: A formula for a sequence that gives the value of a term t_n in terms of the preceding term t_{n-1}. The first term is represented by t_1, the second term in represented by t_2, the third term in represented by t_3, and so forth.

Ex: Find the eighth term of an arithmetic sequence that begins with 1 and 7.

Find the next three terms in each sequence.

a. 4, 9, 14, 19, …

b. 6, -3, 1.5, -.75,…

c. 0, 3, 7, 12, 18, …

d. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$

Ex: If $t_1 = 22$ & $t_n = t_{n-1} - 3$, find the next three terms.

$n=2 \quad t_2 = t_1 - 3 = 19 \qquad n=4 \quad t_4 = t_3 - 3 = 13$

$n=3 \quad t_3 = t_2 - 3 = 16$

Ex: If $t_1 = 3$ and $t_n = t_{n-1} + 4$, find the next five terms in the sequence.

$t_2 = t_1 + 4 = 7 \qquad t_4 = t_3 + 4 = 15$

$t_3 = t_2 + 4 = 11 \qquad t_5 = t_4 + 4 = 19$

Ex: If $t_1 = 64$ and $t_n = \frac{1}{2} t_{n-1}$, find the next four terms.

$t_2 = \frac{1}{2} t_1 = 32 \qquad t_4 = \frac{1}{2} t_3 = 8$

$t_3 = \frac{1}{2} t_2 = 16 \qquad t_5 = \frac{1}{2} t_4 = 4$

Ex: The winner of a contest received \$200 the first year, with a 25% increase over the preceding year's payment for each subsequent year. How much did the contest winner receive during the first 10 years of payments?

$a_{n+1} = 1.25 a_n \qquad S_n = 200\left(\frac{1 - 1.25^{10}}{1 - 1.25}\right) = \$6650.58$

Formula from geometric

Ex: The owners of a certain store reduce the price of their items at the end of each week. If the original price of a blouse is \$250 and its price each week is 4/5 of the previous week, what will be the price of the blouse at the end of the 10th week?

$a_n = 250(r)^n \qquad a_{n+1} = \frac{4}{5} a_n$

$= 250\left(\frac{4}{5}\right)^{10}$

$\approx \$26.85$

Geometric Sequence: the ratio of any term to the previous term is constant.

r = common ratio

Ex: $a_n = 2^n$ 2,4,8,16… Ex: $a_n = \left(\frac{-1}{4}\right)^4$ $\frac{-1}{4}, \frac{1}{16}, \frac{-1}{64}, \frac{1}{256}$..

Finding the nth term of a geometric sequence:

$$a_n = a_1 r^{n-1}$$

Ex: Find the first 5 terms of the geometric sequence whose first term is $a_1 = 4$ and whose ratio is r =3.

4 12 36 108 324

a_1 a_2 a_3 a_4 a_5

Ex: Find the 18th term of the geometric sequence whose first term is 20 and whose common ratio is 1.4.

$a_n = a_1 r^{n-1}$ $a_{18} = 20(1.4)^{17}$

Ex: Write a rule for the nth term. –8, -12, -18, -27, …

$r = \frac{-12}{-8} = 3/2$

$a_n = -8(3/2)^{n-1}$

Ex: $a_4 = 3$, r = 3. Write a rule for the nth term.

1/9 1/3 1 3

a_1 a_2 a_3 a_4

$a_n = 1/9\,(3)^{n-1}$

Geometric Means: the terms between any two nonconsecutive terms of a geometric sequence.

Geometric Mean: $x = \sqrt{ab}$ Recall from Geometry: $\frac{x}{a} = \frac{b}{x}$

Ex: Insert 2 geometric means between 8 and 512.

8, ____, ____, 512

$8r^3 = 512$

$r^3 = 64$

$r = 4$

Geometric series

$$s_n = a_1\left(\frac{1-r^n}{1-r}\right)$$

r = common ratio n = number of terms

works for a finite sequence

$r=5$

Ex: Find the sum of the first 10 terms of the geometric series 1 + 5 + 25 + 125 + 625 …

$S_n = 1\left(\frac{1-5^{10}}{1-5}\right) = 2{,}441{,}406$

Ex: Find the sum of the 1st 8 terms of the geometric series where $a_1 = 8$ and $a_4 = 512$. $r = 4$

$S_n = 8\left(\frac{1-4^8}{1-4}\right) = 174{,}760$

Ex: Given $a_n = 5 \cdot 2^{n-1}$. Find the sum of the first 8 terms.

$r = 2 \quad a_1 = 5$

$S_n = 5\left(\frac{1-2^8}{1-2}\right) =$

Ex: Find $\sum_{n=1}^{10} 4(3)^n$

$r = 3$

$a_1 = 12$

$S_n = 12\left(\frac{1-3^{10}}{1-3}\right)$

Ex: Andrew is investing \$100 monthly in a savings account that pays an APR of 6% compounded monthly. Determine the value of the investment at the end of the year.

Infinite Series: $S = \frac{a_1}{1-r}$

$S_n = 100\left(\frac{1-1.06^{12}}{1-1.06}\right) = 1686.99$

r =common ratio and $|r|<1$ this is when the upper limit is ∞, this only works because r gets close to 0.

Ex: Find $\sum_{n=1}^{\infty} 4(.3)^n$

The first thing we need to do with our calculators is make sure we are in **Sequence Mode**. We do this by going to MODE and changing it to SEQ.

Ex 1: In y = let nMin = 1 and u(n) = 3n +2

Our calculators can aid us in computing sequences and sums of sequences. The **SEQ** feature under **2nd STAT, OPS** #5 and **SUM** feature under **2nd STAT, Math** #5 will help us.

To list the terms of an Arithmetic sequence we can use the SEQ function:

SEQ(expression,variable,begin,end,increment) =

Ex 2: $\sum_{n=1}^{8} 5-3n$

To use this feature type in SUM(SEQ(equation, index, start #, stop #, by 1)

Ex 3: $\sum_{n=1}^{50} 7-3n$ SUM(SEQ($7-3n,n,1,50,1$) =____________________

Ex 4: $\sum_{n=50}^{75} n^2$ ______________________ = ____________________

Ex 5: Find the sum of the first 200 terms of the arithmetic sequence 3,7,11,15

Ex 6: A business makes a \$10,000 profit during its first year. If the yearly profit increases by \$7500 in each subsequent year, what will the profit be in the tenth year? What will be the total profit for the first ten years?