A **sequence** is an ordered list of numbers. Each number in the list is called a **term** of the sequence. The first term of a sequence is denoted as a_1. The second term is denoted as a_2. The term in the nth position is called the nth term and is denoted as a_n. The term before a_n is a_{n-1}.

A sequence is a function whose <u>range</u> is the <u>terms</u> of the sequence and the <u>domain</u> is the <u>position</u> of each term.

There are many types of sequences:

1) An **infinite sequence** is a sequence with an infinite number of terms.
   Can we come up with some examples?

   Ex: 2, 4, 6, 8, 10, 12,….

   Ex: 1, -3, 5, -7, 9, -11, 13 …..

2) A **finite sequence** is a sequence with a finite number of terms. The sequence ends at a certain point.

Three sequences we will learn about:

3) An **Arithmetic sequence** is a sequence in which the difference between each term and the proceeding term is always constant. This difference is known as *d* and is constant term.

4) A **Geometric sequence** is a sequence in which terms are found by multiplying a preceding term by a non-zero constant. This term is known as *r* and is called the common ratio of a geometric sequence.

5) A **Recursive Sequence** is a sequence in which each term is defined using the previous terms.

I. Write a rule for the n^{th} term for the following examples:

6) Ex 1. 1, 2, 3, 4, 5, … Ex. 2 2,4,6,8,10,… Ex.3 2,3,4,5,6,…

7) Ex. 4 5,8,11,14,… Ex. 5 1,3,5,7,9,… Ex. 6 $\frac{2}{5},\frac{2}{25},\frac{2}{125},\frac{2}{625}...$

II. List the first four terms of the sequence given by

Ex: $a_n = 4n - 3$ Ex: $a_n = \frac{(-1)^n}{2n-1}$ Ex: $a_n = 7\left(3^{-n}\right)$

Arithmetic sequences:

The sequence $a_1, a_2, a_3, a_4, \ldots a_n$ is arithmetic if there is a number d such that:

$a_2 - a_1 = d$
$a_3 - a_2 = d$ Where d is the common difference.

Ex: 6, 9, 12, 15, ….3n + 3 The common difference is 3 because 9 – 6 = 3. d =3

Ex: 2, -3, -8, -13, …, 5n + 7 The common difference is –5 because -3–2 = -5. d = -5

<u>Explicit Formula:</u> a formula that defines the n^{th} term.

The n^{th} term of an arithmetic sequence: $\mathbf{a_n = dn + c}$

Is the form for the n^{th} term of an arithmetic sequence. Where d is the common difference and $c = a_1 - d$ (The first term minus the common difference).

Your book represents this differently (** $a_n = a_1 + (n-1)d$ **or** $a_n = a_{n-1} + d$ **)

Ex: Find a formula of an arithmetic sequence whose common difference is 4 and whose first term is 3.

$a_n = dn + c$ We know d = 4. $a_1 = 3$. So c = 3 – 4. c = -1

$a_n = 4n - 1$. The terms of this sequence are: 3, 7, 11, 15, …, 4n – 1.

Ex: Find the formula of the arithmetic sequence whose first term is 3 and whose second term is –1.

$a_n = dn + c$ We know $a_1 = 3$ and $a_2 = -1$. So d = -4. c must be 3–(-4) = 7

$a_n = -4n + 7$

I. Write a rule for the n^{th} term for the following examples:

Ex: 2,4,6,8,10,… Ex: $\frac{2}{5}, \frac{9}{10}, \frac{7}{5}, \frac{19}{10} \ldots$ Ex: 4,8,12,16,…

Ex 1: The fifth term of an arithmetic sequence is 25 and the 12^{th} term is 60. Write the first several terms of this sequence.

$a_5 = 25$ $a_{12} = 60$
$a_{12} = a_5 + 7d$ Where 7 is the difference in the term numbers.
$60 = 25 + 7d$
$35 = 7d$
$5 = d$

Since $a_5 = 25$ we can subtract 5 to get each term in the sequence down to the first.

5, 10, 15, 20, 25

Ex 2: Find the eighth term of an arithmetic sequence that begins with 1 and 7.
d = 7 – 1 = 6

Method 1: Write out the first 8 terms. 1, 7, 13, 19, 25, 31, 37, 43

Method 2: Find the nth term by finding c.
$a_n = dn + c$ $c = a_1 - d$ $c = 1 - 6$ or -5
$a_n = 6n - 5$
So $a_8 - 6(8) - 5 = 43$

Ex 3: Find a rule for the nth term and fill in the missing terms.

____, __4__, _____, _____, ___22_, ______

Ex 4: $a_7 = 34$ $a_{18} = 122$ Write a rule for the nth term.

Arithmetic means: the terms between any two nonconsecutive terms of an arithmetic sequence.

The terms between 2 given terms of an arithmetic sequence are called *arithmetic means*.

10, 13, 16, 19, 22
3 arithmetic means

10, 14, 18, 22
2 arithmetic means

Ex 5. Insert 4 arithmetic means between 15 and 50.

15, _____, _____, _____, _____, 50

Ex 6: Form an arithmetic sequence that has six arithmetic means between -12 & 23.

Summation Notation: the sum of a sequence is also known as an **Arithmetic Series**.

$$\sum_{k=1}^{m} c_k = c_1 + c_2 + c_3 + \ldots + c_m$$

Sigma Notation: the sum of the first n terms of a sequence (called a series)

Ex: $\sum_{i=2}^{5} 4i$

Ex: $\sum_{k=3}^{8} (4-k)$

Partial Sums of an **Arithmetic Sequence**:

$$\mathbf{S_n} = \frac{n}{2}(a_1 + a_n)$$

This means that we add the first and last terms, then multiply by the number of terms divided by 2.

Ex: Find the sum of 2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 + 18 + 20.

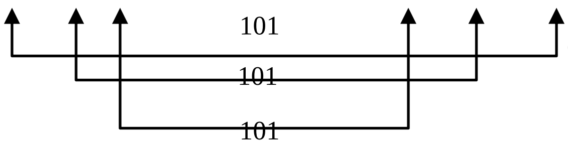
We know that this is an arithmetic sequence because d = 2 and there are 10 terms.

$n = 10$, $a_1 = 2$ and $a_n = 20$

$$S_n = \frac{10}{2}(2 + 20) = 5(22) = 110$$

Gauss: asked to add number from 1 to 100 in 3rd grade.

1 + 2 + 3 + 4 + 5 + … + 98 + 99 + 100



(50 pairs)(101) = 5050 or ______________

Ex: Find the sum of the integers from 1 to 500.

$$S_n = \frac{n}{2}(a_1 + a_n)$$

$n = 500$, $a_1 = 1$ and $a_n = 500$

$$S_n = \frac{500}{2}(1 + 500) = 250(501) = 125,250$$

Ex: Find the sum of the first 200 terms of the arithmetic sequence: 3, 7, 11, 15…..

$$S_n = \frac{n}{2}(a_1 + a_n)$$

$n = 200$, $a_1 = 3$, $a_n = ?$, and $d = 4$

1) Find a_n

   $a_n = dn + c$ Where $c = 3 - 4 = -1$

   $a_n = 4n + -1$

2) Find the 200th term

   $a_{200} = 4(200) - 1 = 800 - 1 = 799$

3) Find S_n

$$S_n = \frac{200}{2}(3 + 799) = 100\ (802) = 80,200$$

Ex: Find the sum of the arithmetic series where $a_1 = -111$, $d = 3$, and $a_n = 9$

Ex: Find the first three terms of the arithmetic series where $a_1 = 10$, $a_n = -46$, and $S_n = -522$.

Recursive Sequences: A formula for a sequence that gives the value of a term t_n in terms of the preceding term t_{n-1}. The first term is represented by t_1, the second term in represented by t_2, the third term in represented by t_3, and so forth.

Ex: Find the eighth term of an arithmetic sequence that begins with 1 and 7.

Find the next three terms in each sequence.

a. 4, 9, 14, 19, …

b. 6, -3, 1.5, -.75,…

c. 0, 3, 7, 12, 18, …

d. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$

Ex: If $t_1 = 22$ & $t_n = t_{n-1} - 3$, find the next three terms.

Ex: If $t_1 = 3$ and $t_n = t_{n-1} + 4$, find the next five terms in the sequence.

Ex: If $t_1 = 64$ and $t_n = \frac{1}{2} t_{n-1}$, find the next four terms.

Ex: The winner of a contest received $200 the first year, with a 25% increase over the preceding year's payment for each subsequent year. How much did the contest winner receive during the first 10 years of payments?

Ex: The owners of a certain store reduce the price of their items at the end of each week. If the original price of a blouse is $250 and its price each week is 4/5 of the previous week, what will be the price of the blouse at the end of the 10th week?

Geometric Sequence: the ratio of any term to the previous term is constant.

r = common ratio

Ex: $a_n = 2^n$ 2,4,8,16… Ex: $a_n = \left(\frac{-1}{4}\right)^4$ $\frac{-1}{4}, \frac{1}{16}, \frac{-1}{64}, \frac{1}{256}..$

Finding the nth term of a geometric sequence:

$$\mathbf{a_n = a_1 r^{n-1}}$$

Ex: Find the first 5 terms of the geometric sequence whose first term is $a_1 = 4$ and whose ratio is r =3.

Ex: Find the 18th term of the geometric sequence whose first term is 20 and whose common ratio is 1.4.

Ex: Write a rule for the nth term. –8, -12, -18, -27, …

Ex: $a_4 = 3$, r = 3. Write a rule for the nth term.

Geometric Means: the terms between any two nonconsecutive terms of a geometric sequence.

Geometric Mean: x = $\sqrt{ab}$ Recall from Geometry: $\frac{x}{a} = \frac{b}{x}$

Ex: Insert 2 geometric means between 8 and 512.

8, _____, _____, 512

Geometric series

$s_n = a_1 \left(\frac{1-r^n}{1-r}\right)$ r = common ratio n = number of terms

works for a finite sequence

Ex: Find the sum of the first 10 terms of the geometric series 1 + 5 + 25 + 125 + 625 …

Ex: Find the sum of the 1st 8 terms of the geometric series where $a_1 = 8$ and $a_4 = 512$.

Ex: Given $a_n = 5 \cdot 2^{n-1}$. Find the sum of the first 8 terms.

Ex: Find $\sum_{n=1}^{10} 4(3)^n$

Ex: Andrew is investing $100 monthly in a savings account that pays an APR of 6% compounded monthly. Determine the value of the investment at the end of the year.

Infinite Series: $S = \dfrac{a_1}{1-r}$

r =common ratio and $|r| < 1$ this is when the upper limit is ∞, this only works because r gets close to 0.

Ex: Find $\sum_{n=1}^{\infty} 4(.3)^n$

The first thing we need to do with our calculators is make sure we are in **Sequence Mode**.
We do this by going to MODE and changing it to SEQ.

Ex 1: In y = let nMin = 1 and u(n) = 3n +2

Our calculators can aid us in computing sequences and sums of sequences. The **SEQ** feature under **2nd STAT**, **OPS** #5 and **SUM** feature under **2nd STAT**, **Math** #5 will help us.

To list the terms of an Arithmetic sequence we can use the SEQ function:
SEQ(expression,variable,begin,end,increment) =

Ex 2: $\sum_{n=1}^{8} 5-3n$

To use this feature type in SUM(SEQ(equation, index, start #, stop #, by 1)

Ex 3: $\sum_{n=1}^{50} 7-3n$ SUM(SEQ(7 – 3*n*,*n*,1,50,1) =___________________

Ex 4: $\sum_{n=50}^{75} n^2$ ______________________ = ____________________

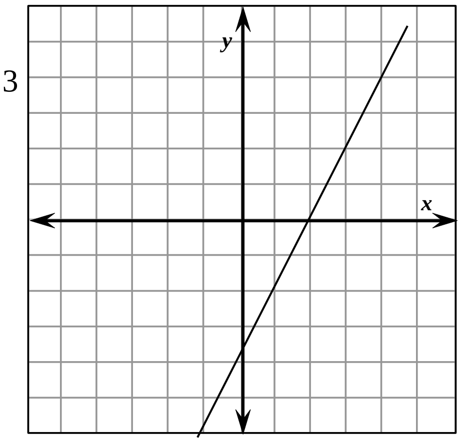
Ex 5: Find the sum of the first 200 terms of the arithmetic sequence 3,7,11,15

Ex 6: A business makes a $10,000 profit during its first year. If the yearly profit increases by $7500 in each subsequent year, what will the profit be in the tenth year? What will be the total profit for the first ten years?

Limit: the y-value as x approaches a certain number from both sides.

$$\lim_{x \to c} f(x) = L$$

Ex: Continuous Graphs:



$\lim_{x \to 3} (2x - 4)$ means "the limit" of 2x – 4 as x approaches 3

Enter it in the calculator and look at the table around 3
It gets closer to 2 as x gets closer to 3.

You can find this by direct substitution: $f(3) = 2(3) - 4$
on continuous graphs.

Ex: Graphs with a discontinuity (a hole).

$$\lim_{x \to 1} \frac{x^3 - x^2 + x + 3}{x + 1}$$ you can't do direct substitution here because you'd get 0 in the denominator.

Enter it into calculator and look at table → gets closer to 2 as x approaches 1.

f(x) does not have to exist at the point for it to have a limit.

Ex: Finding limits from a graph

$$f(x) = \begin{cases} 2x & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases}$$

Can graph using a calculator by
$y_1 = (2x)(x > 1) + (2x)(x < 1)$

Inequalities are found using 2^{nd} MATH

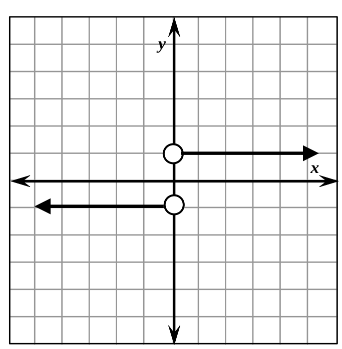
Find the $\lim_{x \to 1} f(x)$

The limit is still 2, because as x gets closer to one, the y-value approaches 2, regardless of the fact that at x = 1, y = 0.

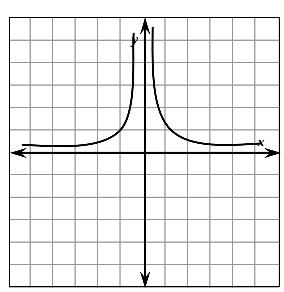
Limits that don't exist: (3 conditions)

1. The limit must approach the same value from both sides, or it does not exist.

   Ex: $\lim_{x \to 0} \frac{|2x|}{2x}$



2. The limit does not approach 1 number, it goes up endlessly, so it does not exist.



Ex: What about $\lim_{x \to -2} \frac{x+3}{x+2}$?

3. Oscillating graph

$\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$

DNE because it is impossible to pick one value between -1 and 1

Direct substitution is the easiest way to find limits, and it will work on any **continuous** graph that does not fit the conditions above.

Direct substitution works on non-continuous graphs if you are finding the limit as x approaches a number that is not the "hole".

Ex: $\lim_{x \to 4} \frac{x^2 - 3x + 2}{x - 8}$

Techniques for Evaluating Limits:

Direct Substitution:

Ex 1. $\lim_{x \to 2} x^2 - 2x + 3$

Ex 2. $\lim_{x \to 2} \frac{x^2 - x + 2}{x + 3}$

Ex 3. $\lim_{x \to 0} \cos x$

You would not be able to do the 2nd example if it asked for $\lim_{x \to -3}$ because the denominator would be 0.

Cancellation Method: For rational functions whose denominator and numerator would be zeros by direct substitution.

Factor and cancel common factors!
Then use Direct Substitution

Ex1. $\lim_{x \to 3} \frac{x^2 - 5x + 6}{x - 3}$

Ex2. $\lim_{x \to -2} \frac{2x^2 - x - 10}{x + 2}$

Rationalization Method: when the cancellation method doesn't work, multiply by the conjugate.

Multiply by the conjugate!
Simplify
Then use Direct Substitution

Ex1. $\lim_{x \to -3} \frac{\sqrt{x+7}-2}{x+3}$ Ex2. $\lim_{x \to 0} \frac{\sqrt{x+1}-1}{x}$

Approximating a Limit: for functions where you can not cancel, rationalize or do direct substitution.

Enter *f*(x) into TI-83
Hit Trace and arrow left and right of the limit
Average the 2 y-values

Ex1. $\lim_{x \to 0} (1+x)^{\frac{1}{x}}$ $\approx \frac{3.0783+2.4760}{2}$ ≈ 2.7771

(-.2128, 3.0783)
(.2128, 2.4760)

Properties of Limits:

Let b and c be real numbers and let n be a positive integer.

a. $\lim_{x \to c} B = B$ b. $\lim_{x \to c} x = c$

c. $\lim_{x \to c} x^n = c^n$ d. $\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$ for n even and $c > 0$

Operations with Limits:

Let b, c be real numbers, let n be a positive integer, and let *f* and *g* be functions with the following limits.

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = K$$

a. Scalar multiple: $\lim_{x \to c} [bf(x)] = bL$ b. Sum or Difference: $\lim_{x \to c} [f(x) \pm g(x)] = L \pm K$

c. Product: $\lim_{x \to c} f(x) \bullet g(x) = LK$ d. . Quotient: $\lim_{x \to c} \left[\frac{f(x)}{g(x)}\right] = \frac{L}{K}$ *provided* $K \neq 0$

e. Power: $\lim_{x \to c} [f(x)^n] = L^n$

Ex: If $\lim_{x\to c} f(x) = 3$ and $\lim_{x\to c} g(x) = -2$ find each of the following:

a. $\lim_{x\to c} [f(x)g(x)]^2 =$

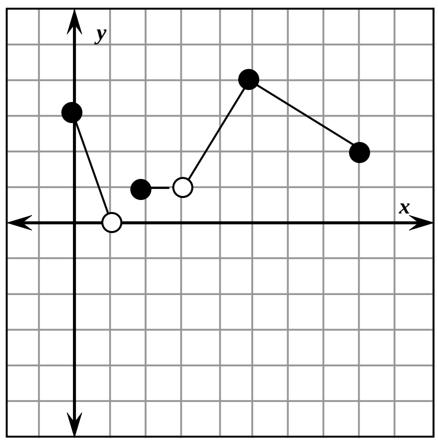
b. $\lim_{x\to c} [6f(x)g(x)] =$

b. $\lim_{x\to c} \frac{5g(x)}{4f(x)} =$

d. $\lim_{x\to c} \frac{1}{\sqrt{f(x)}} =$

Ex: If $f(x) = x^2 - 3x + 4$ $g(x) = 2x - 5$ $\lim_{x\to 2} [f(x) + g(x)] =$

Ex: Let *f*(x) =



Ex: $\lim_{x\to 0^+} f(x) =$

Ex: $\lim_{x\to 2^-} f(x) =$

Ex: $\lim_{x\to 1^+} f(x) =$

Ex: $\lim_{x\to 1^-} f(x) =$

Ex: $\lim_{x\to 3^+} f(x) =$

Ex: $\lim_{x\to 3^-} f(x) =$

Limit of a Difference Quotient:

Find $\lim_{x\to 0} \frac{f(x+h) - f(x)}{h}$ for each of the following functions.

a. $f(x) = x + 2$

b. $g(x) = x^2$

c. $f(x) = x^2 + 3$

d. $g(x) = x^4$