## Extrema

### Absolute Extrema

1. $x = c$ is an absolute maximum of $f(x)$ if $f(c) \geq f(x)$ for all $x$ in the domain.
2. $x = c$ is an absolute minimum of $f(x)$ if $f(c) \leq f(x)$ for all $x$ in the domain.

### Fermat's Theorem

If $f(x)$ has a relative (or local) extrema at $x = c$, then $x = c$ is a critical point of $f(x)$.

### Extreme Value Theorem

If $f(x)$ is continuous on the closed interval $[a,b]$ then there exist numbers $c$ and $d$ so that, **1.** $a \leq c, d \leq b$, **2.** $f(c)$ is the abs. max. in $[a,b]$, **3.** $f(d)$ is the abs. min. in $[a,b]$.

### Finding Absolute Extrema

To find the absolute extrema of the continuous function $f(x)$ on the interval $[a,b]$ use the following process.

1. Find all critical points of $f(x)$ in $[a,b]$.
2. Evaluate $f(x)$ at all points found in Step 1.
3. Evaluate $f(a)$ and $f(b)$.
4. Identify the abs. max. (largest function value) and the abs. min.(smallest function value) from the evaluations in Steps 2 & 3.

### Relative (local) Extrema

1. $x = c$ is a relative (or local) maximum of $f(x)$ if $f(c) \geq f(x)$ for all $x$ near $c$.
2. $x = c$ is a relative (or local) minimum of $f(x)$ if $f(c) \leq f(x)$ for all $x$ near $c$.

### 1st Derivative Test

If $x = c$ is a critical point of $f(x)$ then $x = c$ is

1. a rel. max. of $f(x)$ if $f'(x) > 0$ to the left of $x = c$ and $f'(x) < 0$ to the right of $x = c$.
2. a rel. min. of $f(x)$ if $f'(x) < 0$ to the left of $x = c$ and $f'(x) > 0$ to the right of $x = c$.
3. not a relative extrema of $f(x)$ if $f'(x)$ is the same sign on both sides of $x = c$.

### 2nd Derivative Test

If $x = c$ is a critical point of $f(x)$ such that $f'(c) = 0$ then $x = c$

1. is a relative maximum of $f(x)$ if $f''(c) < 0$.
2. is a relative minimum of $f(x)$ if $f''(c) > 0$.
3. may be a relative maximum, relative minimum, or neither if $f''(c) = 0$.

### Finding Relative Extrema and/or Classify Critical Points

1. Find all critical points of $f(x)$.
2. Use the 1st derivative test or the 2nd derivative test on each critical point.

### Mean Value Theorem

If $f(x)$ is continuous on the closed interval $[a,b]$ and differentiable on the open interval $(a,b)$ then there is a number $a < c < b$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

### Newton's Method

If $x_n$ is the $n^{\text{th}}$ guess for the root/solution of $f(x) = 0$ then $(n+1)^{\text{st}}$ guess is $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ provided $f'(x_n)$ exists.

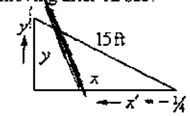





## Related Rates

Sketch picture and identify known/unknown quantities. Write down equation relating quantities and differentiate with respect to $t$ using implicit differentiation (*i.e.* add on a derivative every time you differentiate a function of $t$). Plug in known quantities and solve for the unknown quantity.

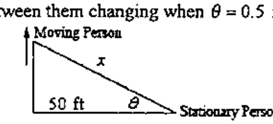
**Ex.** A 15 foot ladder is resting against a wall. The bottom is initially 10 ft away and is being pushed towards the wall at $\frac{1}{4}$ ft/sec. How fast is the top moving after 12 sec?




$x'$ is negative because $x$ is decreasing. Using Pythagorean Theorem and differentiating,
$x^2 + y^2 = 15^2 \Rightarrow 2xx' + 2yy' = 0$
After 12 sec we have $x = 10 - 12\left(\frac{1}{4}\right) = 7$ and so $y = \sqrt{15^2 - 7^2} = \sqrt{176}$. Plug in and solve for $y'$.

$7\left(-\frac{1}{4}\right) + \sqrt{176}\, y' = 0 \Rightarrow y' = \dfrac{7}{4\sqrt{176}}$ ft/sec

**Ex.** Two people are 50 ft apart when one starts walking north. The angle $\theta$ changes at 0.01 rad/min. At what rate is the distance between them changing when $\theta = 0.5$ rad?




We have $\theta' = 0.01$ rad/min. and want to find $x'$. We can use various trig fcns but easiest is,

$\sec\theta = \dfrac{x}{50} \Rightarrow \sec\theta\tan\theta\,\theta' = \dfrac{x'}{50}$

We know $\theta = 0.05$ so plug in $\theta'$ and solve.

$\sec(0.5)\tan(0.5)(0.01) = \dfrac{x'}{50}$

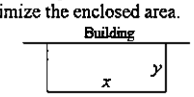
$x' = 0.3112$ ft/sec

Remember to have calculator in radians!

## Optimization

Sketch picture if needed, write down equation to be optimized and constraint. Solve constraint for one of the two variables and plug into first equation. Find critical points of equation in range of variables and verify that they are min/max as needed.

**Ex.** We're enclosing a rectangular field with 500 ft of fence material and one side of the field is a building. Determine dimensions that will maximize the enclosed area.

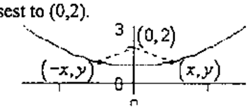



Maximize $A = xy$ subject to constraint of $x + 2y = 500$. Solve constraint for $x$ and plug into area.

$x = 500 - 2y \Rightarrow \begin{aligned} A &= y(500 - 2y) \\ &= 500y - 2y^2 \end{aligned}$

Differentiate and find critical point(s).
$A' = 500 - 4y \Rightarrow y = 125$
By 2nd deriv. test this is a rel. max. and so is the answer we're after. Finally, find $x$.
$x = 500 - 2(125) = 250$
The dimensions are then 250 x 125.

**Ex.** Determine point(s) on $y = x^2 + 1$ that are closest to (0,2).




Minimize $f = d^2 = (x-0)^2 + (y-2)^2$ and the constraint is $y = x^2 + 1$. Solve constraint for $x^2$ and plug into the function.

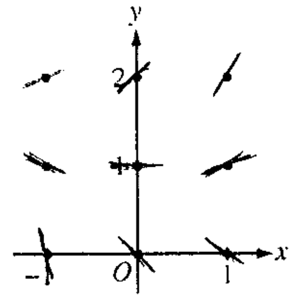
$x^2 = y - 1 \Rightarrow f = x^2 + (y-2)^2$
$= y - 1 + (y-2)^2 = y^2 - 3y + 3$

Differentiate and find critical point(s).
$f' = 2y - 3 \Rightarrow y = \frac{3}{2}$
By the 2nd derivative test this is a rel. min. and so all we need to do is find $x$ value(s).
$x^2 = \frac{3}{2} - 1 = \frac{1}{2} \Rightarrow x = \pm\frac{1}{\sqrt{2}}$
The 2 points are then $\left(\frac{1}{\sqrt{2}}, \frac{3}{2}\right)$ and $\left(-\frac{1}{\sqrt{2}}, \frac{3}{2}\right)$.

# AP® CALCULUS AB
# 2007 SCORING GUIDELINES (Form B)

## Question 5

Consider the differential equation $\frac{dy}{dx} = \frac{1}{2}x + y - 1$.




(a) On the axes provided, sketch a slope field for the given differential equation at the nine points indicated.
**(Note: Use the axes provided in the exam booklet.)**

(b) Find $\frac{d^2y}{dx^2}$ in terms of $x$ and $y$. Describe the region in the $xy$-plane in which all solution curves to the differential equation are concave up.

(c) Let $y = f(x)$ be a particular solution to the differential equation with the initial condition $f(0) = 1$. Does $f$ have a relative minimum, a relative maximum, or neither at $x = 0$? Justify your answer.

(d) Find the values of the constants $m$ and $b$, for which $y = mx + b$ is a solution to the differential equation.

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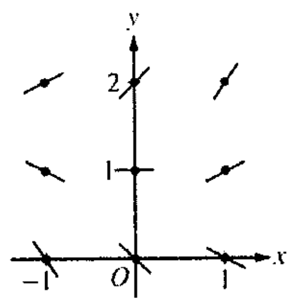
(a)




2 : Sign of slope at each point and relative steepness of slope lines in rows and columns.

(b) $\frac{d^2y}{dx^2} = \frac{1}{2} + \frac{dy}{dx} = \frac{1}{2}x + y - \frac{1}{2}$

Solution curves will be concave up on the half-plane above the line $y = -\frac{1}{2}x + \frac{1}{2}$.

3 : $\begin{cases} 2 : \frac{d^2y}{dx^2} \\ 1 : \text{description} \end{cases}$

(c) $\left.\frac{dy}{dx}\right|_{(0,\,1)} = 0 + 1 - 1 = 0$ and $\left.\frac{d^2y}{dx^2}\right|_{(0,\,1)} = 0 + 1 - \frac{1}{2} > 0$

Thus, $f$ has a relative minimum at $(0, 1)$.

2 : $\begin{cases} 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

(d) Substituting $y = mx + b$ into the differential equation:

$m = \frac{1}{2}x + (mx + b) - 1 = \left(m + \frac{1}{2}\right)x + (b - 1)$

Then $0 = m + \frac{1}{2}$ and $m = b - 1$: $m = -\frac{1}{2}$ and $b = \frac{1}{2}$.

2 : $\begin{cases} 1 : \text{value for } m \\ 1 : \text{value for } b \end{cases}$

### Question 5

At the beginning of 2010, a landfill contained 1400 tons of solid waste. The increasing function $W$ models the total amount of solid waste stored at the landfill. Planners estimate that $W$ will satisfy the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ for the next 20 years. $W$ is measured in tons, and $t$ is measured in years from the start of 2010.

(a) Use the line tangent to the graph of $W$ at $t = 0$ to approximate the amount of solid waste that the landfill contains at the end of the first 3 months of 2010 (time $t = \frac{1}{4}$).

(b) Find $\frac{d^2W}{dt^2}$ in terms of $W$. Use $\frac{d^2W}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the amount of solid waste that the landfill contains at time $t = \frac{1}{4}$.

(c) Find the particular solution $W = W(t)$ to the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ with initial condition $W(0) = 1400$.

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(a) $\left.\frac{dW}{dt}\right|_{t=0} = \frac{1}{25}(W(0) - 300) = \frac{1}{25}(1400 - 300) = 44$

The tangent line is $y = 1400 + 44t$.

$W\left(\frac{1}{4}\right) \approx 1400 + 44\left(\frac{1}{4}\right) = 1411$ tons

$2 : \begin{cases} 1 : \frac{dW}{dt} \text{ at } t = 0 \\ 1 : \text{answer} \end{cases}$

(b) $\frac{d^2W}{dt^2} = \frac{1}{25}\frac{dW}{dt} = \frac{1}{625}(W - 300)$ and $W \geq 1400$

Therefore $\frac{d^2W}{dt^2} > 0$ on the interval $0 \leq t \leq \frac{1}{4}$.

The answer in part (a) is an underestimate.

$2 : \begin{cases} 1 : \frac{d^2W}{dt^2} \\ 1 : \text{answer with reason} \end{cases}$

(c) $\frac{dW}{dt} = \frac{1}{25}(W - 300)$

$\int \frac{1}{W - 300}\, dW = \int \frac{1}{25}\, dt$

$\ln|W - 300| = \frac{1}{25}t + C$

$\ln(1400 - 300) = \frac{1}{25}(0) + C \Rightarrow \ln(1100) = C$

$W - 300 = 1100e^{\frac{1}{25}t}$

$W(t) = 300 + 1100e^{\frac{1}{25}t},\ 0 \leq t \leq 20$

$5 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } W \end{cases}$

Note: max 2/5 [1-1-0-0-0] if no constant of integration

Note: 0/5 if no separation of variables

## Question 6

Solutions to the differential equation $\frac{dy}{dx} = xy^3$ also satisfy $\frac{d^2y}{dx^2} = y^3\left(1 + 3x^2y^2\right)$. Let $y = f(x)$ be a particular solution to the differential equation $\frac{dy}{dx} = xy^3$ with $f(1) = 2$.

(a) Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 1$.

(b) Use the tangent line equation from part (a) to approximate $f(1.1)$. Given that $f(x) > 0$ for $1 < x < 1.1$, is the approximation for $f(1.1)$ greater than or less than $f(1.1)$? Explain your reasoning.

(c) Find the particular solution $y = f(x)$ with initial condition $f(1) = 2$.

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(a) $f'(1) = \left.\frac{dy}{dx}\right|_{(1,\,2)} = 8$

An equation of the tangent line is $y = 2 + 8(x - 1)$.

$2 : \begin{cases} 1 : f'(1) \\ 1 : \text{answer} \end{cases}$

(b) $f(1.1) \approx 2.8$

Since $y = f(x) > 0$ on the interval $1 \le x < 1.1$,

$\frac{d^2y}{dx^2} = y^3\left(1 + 3x^2y^2\right) > 0$ on this interval.

Therefore on the interval $1 < x < 1.1$, the line tangent to the graph of $y = f(x)$ at $x = 1$ lies below the curve and the approximation 2.8 is less than $f(1.1)$.

$2 : \begin{cases} 1 : \text{approximation} \\ 1 : \text{conclusion with explanation} \end{cases}$

(c) $\frac{dy}{dx} = xy^3$

$\int \frac{1}{y^3}\,dy = \int x\,dx$

$-\frac{1}{2y^2} = \frac{x^2}{2} + C$

$-\frac{1}{2 \cdot 2^2} = \frac{1^2}{2} + C \Rightarrow C = -\frac{5}{8}$

$y^2 = \frac{1}{\frac{5}{4} - x^2}$

$f(x) = \frac{2}{\sqrt{5 - 4x^2}}, \quad \frac{-\sqrt{5}}{2} < x < \frac{\sqrt{5}}{2}$

$5 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } y \end{cases}$

Note: max $2/5$ [1-1-0-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

Question 6.

$$\frac{dy}{y^3} = x\,dx$$

c) $\int y^{-3+1}\,dy = \int x\,dx$

$$-\frac{1}{2}y^{-2} = \frac{1}{2}x^2 + C$$

$$-\frac{1}{2y^2} = \frac{1}{2}x^2 + C$$

$$-\frac{1}{8} = \frac{1}{2} + C$$

$$C = -5/8$$

$$\frac{-1}{2y^2} = \frac{1}{2}x^2 - \frac{5}{8}$$

$$\frac{1}{y^2} = -x^2 + \frac{5}{4}$$

$$\frac{1}{y^2} = \frac{-4x^2+5}{4}$$

$$y^2(-4x^2+5) = 4$$

$$y^2 = \frac{4}{5-4x^2}$$

$$y = \frac{2}{\sqrt{5-4x^2}}$$

Question 5

b) $\frac{d^2y}{dx^2} = \frac{1}{2}x + y - 1$

$= 1/2 + \frac{dy}{dx}$

$= \frac{1}{2} + \frac{1}{2}x + y - 1$

$= \frac{1}{2}x + y - \frac{1}{2}$

$y = -1/2x + 1/2$

c) $(0,1)$

$f'' > 0$ minimum

d) $\frac{dy}{dx} = \frac{1}{2}x + mx + b - 1$

$\frac{dy}{dx} = x(\frac{1}{2}+m) + b - 1$

$\frac{1}{2} + m = 0$

$m = -1/2$

$m = b - 1 \quad -1/2 = b - 1$

$b = 1/2$

$$\frac{25\,dW}{W-300} = dt \qquad \frac{dW}{dt} = \frac{1}{25}(W(0) - 300)$$

Question 5 2011

$$\frac{dW}{dt} = \frac{1}{25}(1100) = 44$$

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$$y = 44t + 1400$$

$$W(1/4) = 44(1/4) + 1400 \approx 1411 \text{ tons}$$

$$\frac{d^2W}{dt^2} = \frac{1}{25}W \cdot \frac{dW}{dt}$$

$$= \frac{1}{25}\left[\frac{1}{25}(W - 300)\right]$$

$$= \frac{1}{625}(W - 300) \quad \text{and } W \geq 1400$$

$$W \geq 1400 \quad = \quad \frac{d^2W}{dt^2} > 0 \qquad 0 \leq t \leq 1/4$$

underestimate.

$$25(W-300)^{-1}$$

$$\int \frac{1}{W-300}\,dW = \int \frac{1}{25}\,dt$$

$u = W - 300$

$du = 1$

$$\ln|W - 300| = \frac{1}{25}t + C \qquad \ln(W-300) =$$

$$\ln 1100 = C$$

$$\ln|W - 300| = 1/25\,t + \ln 1100$$

$$W - 300 = e^{1/25\,t + \ln 1100}$$

$$W - 300 = 1100e^{1/25\,t}$$

$$W = 1100e^{1/25\,t} + 300$$