Applications of the Derivative: Optimization

There are the same types of word problems that you encountered in Algebra but they may involve more complicated functions, and our method for solving is much easier.

Strategy:

1. READ THE WHOLE PROBLEM
2. Draw a picture and/ or collect the given info
3. Write down the equation of the quantity to be maximized or minimized.
4. Express $f(x)$ in terms of just one variable
5. Find the domain from the physical restrictions of the problem that is appropriate
6. Find $f'(x)$ and find the critical points from it. (Be sure to check where $f'(x)$ DNE)

   Use either the first or second derivative test to locate the max/min and remember to check the endpoints where appropriate.

1) Find the dimensions of the rectangle with perimeter of 100 cm. and area as large as possible.

$2l + 2w = 100$ $\quad$ $A = -l^2 + 50l$

$2w = -2l + 100$ $\quad$ $dA = -2l + 50$

$w = -l + 50$ $\quad$ $0 = -2l + 50$

$l = 25 \quad w = 25$

2) An open top box is to be made from a piece of cardboard 16 in. by 30 in., by cutting out squares from each corner and bending up the edges. What size should the squares be in order to produce the box with a maximum volume?

$V = (16-2x)(30-2x)\cdot x$ $\quad$ $dV = 3x^2 - 46x + 120$

$= (480 - 92x + 4x^2)\,x$ $\quad$ $0 = (3x - 10)(x - 12)$

$V = 4x^3 - 92x^2 + 480x$ $\quad$ $x = 10/3,\ x = 12$

$dV = 12x^2 - 184x + 480$ $\quad$ Domain $0 < x < 8$

3) Find the radius and height of the largest right circular cylinder that can be inscribed in a right circular cone with base radius 6 in. and height 10 in.

$V = \pi r^2 h$ $\quad$ $\frac{r}{6} = \frac{10-h}{10}$ $\quad$ $10r = 60 - 6h$

$= \pi r^2\left(\frac{5r-30}{-3}\right)$ $\quad$ $\frac{10r-60}{-6} = h$

$= -\frac{5\pi r^3}{3} + 10\pi r^2$ $\quad$ $\frac{5r-30}{-3} = h$

$dV = -5\pi r^2 + 20\pi r$ $\quad$ $r = 4$

$0 = -5\pi r(r-4)$ $\quad$ $h = 10/3$

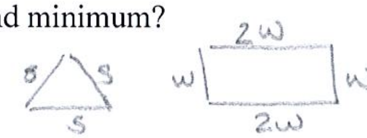
4) A metal cylindrical container with an open top is to hold one cubic foot. If there is no waste in construction, find the dimensions which require the least amount of material.

$r = \frac{1}{\sqrt[3]{\pi}}$ $\quad$ $V = \pi r^2 h$ $\quad$ S.A. $= \pi r^2 + 2\pi r h$

$h = \frac{1}{\sqrt[3]{\pi}}$ $\quad$ $1\,ft^3 = \pi r^2 h$ $\quad$ $= \pi r^2 + 2\pi r\left(\frac{1}{\pi r^2}\right)$

$\frac{1}{\pi r^2} = h$ $\quad$ S.A. $= \pi r^2 + \frac{2}{r}$

5) A wire, 36 cm long, is to be cut into two pieces (or not). One of the pieces is to be bent into an equilateral triangle, and the other into a rectangle whose length is twice its width. Where should the wire be cut so the combined area is maximum? And minimum?



$\max_{\triangle}$ $36\sqrt{3}$

$\max_{\square}$ 72

$A = 2w^2 + \frac{\sqrt{3}}{4}s^2$

$A = 2w^2 + \frac{\sqrt{3}}{4}(12-2w)^2$

$A = 2w^2 + 36\sqrt{3} - 12\sqrt{3}w + \sqrt{3}w^2$

$\frac{dA}{dt} = 4w - 12\sqrt{3} + 2\sqrt{3}w$

$w = \frac{12\sqrt{3}}{4+2\sqrt{3}} = 12\sqrt{3} - 18$

$36 = 3s + 6w$

$36 - 3s = 6w$

$6 - \frac{1}{2}s = w$

$s = -24\sqrt{3} + 48$

$36 - 6w = 3s$

$12 - 2w = s$ ✓

6) A package can be sent by UPS if the sum of its length and its girth (the perimeter of its base) is not more than 96 inches. Find the dimensions of the box of maximum volume that can be sent, if the base of the box is a square.

$4x + y \le 96$

$y = -4x + 96$

Volume $= x^2(-4x+96)$

$= -4x^3 + 96x^2$

$dV = -12x^2 + 192x$

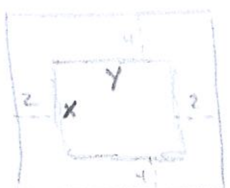
$dV = -12x(x - 16)$

$x = 16$

dimensions $16 \times 16 \times 32$

Notes Optimization Day 2: More Examples 2016

1) You are designing a rectangular poster to contain 50 sq. inches of print with a 4 inch margin at the top and bottom and a 2 inch margin at each side. What overall dimensions will minimize the total amount of paper used?



$50 = x \cdot y$ $\quad y = 50/x$

$A = (x+8)(50/x + 4)$

$A = \frac{400}{x} + 4x + 82$

$dA = -400x^{-2} + 4$

$dA = \frac{-400}{x^2} + 4$

$\frac{4x^2 - 400}{x^2} = \frac{4(x+10)(x-10)}{x^2}$

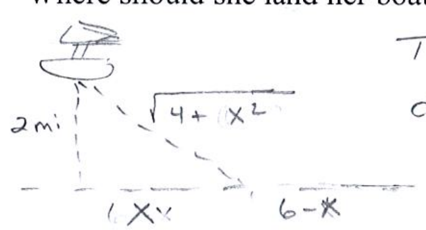
$x = 10$

0 — + 10

Domain $(0, \infty)$

Dimension 18×9

2) Lisa is 2 miles offshore in a boat and wishes to reach a coastal village 6 miles down a straight shoreline from the point on the shore nearest to the boat. She can row 2 mph and walk 5 mph. Where should she land her boat on shore in order to reach the village in the least amount of time?



$T = \frac{\sqrt{4+x^2}}{2} + \frac{(6-x)}{5}$

$dT = \frac{1}{4}(x^2+4)^{-1/2} \cdot 2x - 1/5 \Rightarrow \frac{1}{5} = \frac{x}{2\sqrt{x^2+4}}$

$5x = 2\sqrt{x^2+4}$

$\frac{25}{4}x^2 = x^2 + 4$

$0 = \frac{21x^2}{4} - 4$

$x = \sqrt{16/21}$ $\quad x \approx .873$

3) There are 50 apple trees in an orchard. Each tree produces 800 apples. For each additional tree planted in the orchard, the output per tree drops by 10 apples. How many trees should be added to the existing orchard in order to maximize the total output of trees ?

$P = (50 + x)(800 - 10x)$

$P = 40000 + 300x - 10x^2$

$dP = -20x + 300$

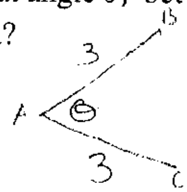
$0 = -20(x - 15)$

$x = 15$

+ — 15

15 trees

4) What angle θ, between two edges of length 3, will result in an isosceles triangle with the largest area?



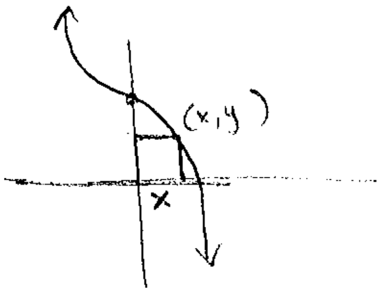
$A = \frac{1}{2}(3)(3)\sin\theta$

$A = \frac{9}{2}\sin\theta$

$dA = \frac{9}{2}\cos\theta$

$\theta = \frac{\pi}{2}$

5) Find the dimensions of the rectangle of largest area which can be inscribed in the closed region bounded by the x-axis, y-axis, and graph of $y = 8 - x^3$.



$A = x \cdot (8 - x^3)$

$A = -x^4 + 8x$

$dA = -4x^3 + 8$

$dA = -4(x^3 - 2)$

$x = \sqrt[3]{2}$

Dimensions $\sqrt[3]{2} \times 6$