Day 3:

Definition of Increasing and Decreasing Functions

A function f is increasing on an interval if for any two numbers a and b in the interval, a < b implies $f(a) < f(b)$.

A function f is decreasing on an interval if for any two numbers a and b in the interval, a < b implies $f(a) > f(b)$.

Let f be a function that is continuous on the closed $[a, b]$ and differentiable on the open interval (a, b).

1. If $f'(x) > 0$ for all x in (a, b), then f is increasing on $[a, b]$.
2. If $f'(x) < 0$ for all x in (a, b), then f is decreasing on $[a, b]$.
3. If $f'(x) = 0$ for all x in (a, b), then f is constant on $[a, b]$.

Ex 1: Find the open intervals on which $f(x) = x^3 - \frac{3}{2}x^2$ is increasing or decreasing. (Graph)

$f'(x) = 3x^2 - 3x$

$3x(x-1)$

Critical Points $x=0$ $x=1$

$f'(x)$ + − +

-1 0 $1/2$ 1 2

$f(x)$ is increasing $(-\infty,0) \cup (1,\infty)$

decreasing $(0,1)$

Ex 2: Find the open intervals on which $f(x) = \dfrac{x^2}{4x+4}$ is increasing or decreasing.

$x \neq -1$

$f'(x) = \dfrac{(4x+4)2x - x^2 \cdot 4}{(4x+4)^2} = \dfrac{4x(x+2)}{(4x+4)^2} = 0$

inc $(-\infty,-2) \cup (0,\infty)$

dec $(-2,-1) \cup (-1,0)$

Critical Points. $x = -2, -1, 0$

+ − − +

-3 -2 -1.5 -1 $-1/2$ 0 4

Guidelines for Finding Intervals on Which a Function is Increasing or Decreasing

1. Locate the Critical Points on the interval (a, b), use these numbers to determine test intervals.
2. Determine the sign of $f'(x)$ at one test value in each of the intervals.
3. Determine whether f is increasing or decreasing.

The First Derivative Test:

> Let c be a critical number of a function f that is continuous on an open interval I containing c.
>
> If f is differentiable on the interval, except possibly at c, then $f(c)$ can be classified as follows.
>
> 1. If $f'(x)$ changes from negative to positive at c, then $f(c)$ is a relative minimum of f.
>
> 2. If $f'(x)$ changes from positive to negative at c, then $f(c)$ is a relative maximum of f.
>
> 3. If $f'(x)$ does not change sign at c, then $f(c)$ is neither a relative minimum nor a relative maximum.

Find the relative extrema of the functions below, if they exist.

Ex 3: $f(x) = -x^3 - 3x^2 - 1$

$f'(x) = -3x(x+2)$

Critical Points $x=0$ $x=-2$

$f'(x)$: $-$ | $+$ | $-$

$-3 \quad -2 \quad -1 \quad 0 \quad 1$

decreasing $(-\infty,-2) \cup (0,\infty)$

increasing $(-2,0)$

Relative min $(-2,-5)$ max $(0,-1)$

Ex 4: $f(x) = \dfrac{-2}{x^2-4}$

Critical Points $x \neq \pm 2$

$f'(x) = -\dfrac{-2(2x)}{(x^2-4)^2}$

$x = 0$

$f'(x)$: $-$ | $-$ | $+$ | $+$

$-3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3$

decreasing $(-\infty,-2) \cup (-2,0)$

increasing $(0,2) \cup (2,\infty)$

Relative min $(0, 1/2)$

Ex 5: $f(x) = x - 2\sin(x)$; $(0, 2\pi)$

$f'(x) = 1 - 2\cos(x)$

$\cos x = 1/2$

Critical Points $x = \pi/3, 5\pi/3$

$f'(x)$: $-$ | $+$ | $-$

$\pi/6 \quad \pi/3 \quad \pi \quad 5\pi/3 \quad 11\pi/6$

Relative min $\left(\pi/3, \frac{\pi}{3} - \sqrt{3}\right)$ max $\left(5\pi/3, \frac{5\pi}{3} + \sqrt{3}\right)$

Ex 6: $f(x) = (x^2-4)^{\frac{1}{3}}$

$f'(x) = \frac{1}{3}(x^2-4)^{-2/3} \cdot 2x$

$= \dfrac{2x}{3\sqrt[3]{(x^2-4)^2}}$

Critical Points $x=0$ $x=\pm 2$

$f'(x)$: $-$ | $-$ | $+$ | $+$

$-3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3$

Relative min $(0, -4^{1/3})$

Ex 7: $f(x) = \cos^2(x) - 2\sin(x)$; $(0, 2\pi)$

$f'(x) = 2\cos(x) \cdot (-\sin x) - 2\cos x$

$= 2\cos(x)(-\sin x - 1)$

Critical Points $x = \pi/2, 3\pi/2$

$-$ | $+$ | $-$

$\pi/4 \quad \pi/2 \quad \pi \quad 3\pi/2 \quad 7\pi/4$

relative min $(\pi/2, -2)$ relative max $(3\pi/2, 2)$

Ex 8: $f(x) = x^4 - 4x^3$

$f'(x) = 4x^3 - 12x^2$

$4x^2(x-3)$

Critical Points $x=0$ $x=3$

$-$ | $-$ | $+$

$0 \quad 3$

Relative Min $(3,-27)$