Expected Value and Fairness Notes

Expected Value:

To calculate the expected value, you multiply the probability of each outcome by the possible outcomes.

$$E(x)=x_1p_1+x_2p_2+\ldots\ldots+x_np_n$$

1. The daily earnings of an employee who works on a commission basis are given by the following probability distribution. Find the employee's expected earnings.

x (in \$)	0	25	50	75	100	125	150
P(x)	0.07	.12	.17	.14	.28	.18	.04

0 + 3 + 8.5 + 10.5 + 28 + 22.5 + 6 = \$78.5

2. Consider the experiment of rolling two dice and adding the numbers on top of the faces. Calculate the expected value of the probability distribution.

~~11~~ 12 13 14 15 16
21 22 23 24 25 26
31 32 33 34 35 36
41 42 43 44 45 46
51 52 53 54 55 56
61 62 63 64 65 66.

2	3	4	5	6	7	8	9	10	11	12
1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

$\frac{2}{36}+\frac{6}{36}+\frac{12}{36}+\frac{20}{36}+\frac{30}{36}+\frac{42}{36}+\frac{40}{36}+\frac{36}{36}+\frac{30}{36}+\frac{22}{36}+\frac{12}{36}$

$=\frac{252}{36}=7$

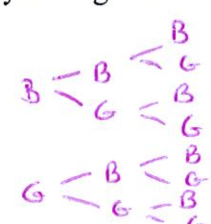
3. If the sum of two rolled dice is 8 or more, player A wins \$2; if not, you lose \$1. Find the expected value of the game for you.

#1 - 7	8 - 12
(-1) 21/36	15/36 (2)

$-21/36 + 30/36 = 9/36 = .25$ ¢

4. Consider a family with 3 children. Assume that births of boys and girls are equally likely. Find the expected value for the number of girls.

0	1	2	3
1/8	3/8	3/8	1/8



$0 + 3/8 + 6/8 + 3/8 = 1.5$

5. Consider a family with 5 children. Assume that births of boys and girls are equally likely. Find the expected value for the number of boys.

BBBBB BBBBG BBBGB BBBGG
BBGBB BBGBG BBGGB BBGGG
BGBBB BGBBG BGBGB BGBGG
BGGBB BGGBG BGGGB BGGGG

0	1	2	3	4	5
1/32	5/32	10/32	10/32	5/32	1/32

$\frac{80}{32} = 2.5$

${}_5C_1 \left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^4$

${}_5C_2 \left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3$

${}_5C_3 \left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2$

${}_5C_4 \left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^1$

In a lottery, the value of a ticket is a random variable, defined to be the amount of money you win less the cost of playing.

6. Suppose that in a lottery for charity with 225 tickets, each ticket costs \$1. First prize is \$50, second prize \$30, and third prize is \$20. Then the possible values of the random variable are \$49, \$29, \$19, and \$-1.

   a. Why is one of the values negative?

   Pay for the ticket and they lost.

   b. The probability of winning first prize is $\frac{1}{225}$. The same probability holds for second and third prizes. Find the probability of winning nothing.

   $\frac{1}{225}$ 1st $\quad \frac{1}{225}$ 2nd $\quad \frac{1}{225}$ 3rd. $\quad \boxed{\frac{222}{225}}$

   c. Find the expected value of a ticket.

   $$49\left(\frac{1}{225}\right) + 29\left(\frac{1}{225}\right) + 19\left(\frac{1}{225}\right) - 1\left(\frac{228}{225}\right) = \frac{-125}{225} \text{ or } -.56¢$$

7. The Island Club is holding a fund-raising raffle. Ten thousand tickets have been sold for \$2 each. There will be a first prize of \$3000, 3 second prizes of \$1000 each, 5 third prizes of \$500 each, and 20 consolation prizes of \$100 each.. Letting X denote the net winnings associated with the tickets, find E(x). Interpret your result. (Note: net winnings is the amount won after the cost of the ticket)

   2998 $\quad$ 998 $\quad$ 498 $\quad$ 98 $\quad$ -2

   $\frac{1}{10,000} \quad \frac{3}{10,000} \quad \frac{5}{10,000} \quad \frac{20}{10,000} \quad \frac{9971}{10,000}$

   $$2,998 + 2,994 + 2,490 + 1,960 - 19,942 = \frac{-9500}{10,000} = -.95$$

8. In a lottery, 120 tickets are sold at \$1 each. First prize is \$50 and second is \$20. Find the expected value of a ticket.

   49 $\quad$ 19 $\quad$ -1

   $\frac{1}{120} \quad \frac{1}{120} \quad \frac{118}{120} \quad = -.42¢$

9. In a certain state's lottery, six numbers are randomly chosen without repetition from the numbers 1 to 40. If you correctly pick all 6 numbers, only 5 of the 6, or only 4 of the 6, then you will \$1 million, \$1000 or \$100, respectively. What is the value of a \$1 lottery ticket?

   $\frac{{}_6C_6}{{}_{40}C_6}$ (1) $\quad \frac{{}_6C_5\,{}_{34}C_1}{{}_{40}C_6}$ (204) $\quad \frac{{}_6C_4\,{}_{34}C_2}{{}_{40}C_6}$ (8415) $\quad$ Loser ${}_{40}C_6 - 8620$ (3829760)

   999,999 $\quad$ 999 $\quad$ 99 $\quad$ -1

   $$= \frac{999,999 + 203,796 + 833085 - 3829760}{{}_{40}C_6} = -.467$$

10. In the game of roulette as played in Las Vegas casinos, the wheel is divided into 38 compartments numbered 1-36, 0, and 00. One half of the numbers 1-36 are red, the other half are black, and 0 and 00 are green. Of the many types of bets that may be placed, one type involves betting on the outcome of the color of the winning number. For example, one may place a certain sum of money on red. If the winning number is red, one wins an amount equal to the bet placed and losses the bet otherwise. Find the expected value of the winnings on a \$1 best placed on red.

    red $\quad$ black $\quad$ Green

    18/38 $\quad$ 18/38 $\quad$ 2/38

    $$1(18/38) - 1(18/38) - 1(2/38) = \frac{-2}{38} = -.0526$$

Fairness in a Game:

For a game to be fair, no one has advantage of another player. This translates to having an expected value of 0. So to determine if something is fair, calculate the expected value.

11. Most game of chance are not fair but people participate anyway.

12. If the sum of two rolled dice is 8 or more, you win \$2; if not, you lost \$1.

a. Show that this is not a fair game.

$$-\frac{21}{36} + \frac{30}{36} = \frac{9}{36} = .25¢$$

b. To have a fair game, the \$2 winnings should instead be what amount?

$$-\frac{21}{36} + x\left(\frac{15}{36}\right) = 0 \qquad 15x - 21 = 0 \qquad 15x = 21 \qquad x = 21/15 \text{ or } 1.40$$

13. Two coins are tossed. If both land heads up, then player A wins \$4 from player B. If exactly one coin lands heads up, then B wins \$1 from A. If both land tails up, then B wins \$2 from A. Is this a fair game?

HH	HT/TH	TT
1/4	1/2	1/4

H < H, T
T < H, T

$$4(1/4) - 1(1/2) - 2(1/4) = 0$$

14. Two dice are rolled. If the sum of the numbers showing on the dice is odd, player A wins \$1 from player B. If both dice show the same number, A wins \$3 from B. Otherwise B wins \$3 from A. Is this a fair game?

odd	same	otherwise.
1/2	1/6	1/3

$$1(1/2) + 3(1/6) - 3(1/3) = 0$$

15. Mike and Bill play a card game with a standard deck of 52 cards. Mike selects a card from a well-shuffled deck and receives A dollars from Bill if the card selected is a diamond; otherwise, Mike pays Bill a dollar. Determine the value of A if the game is to be fair.

diamond	otherwise
1/4	3/4

$$A(1/4) - 1(3/4) = 0 \qquad A = \$3$$

16. Latoya and Richard play a dice game with two die. Latoya rolls the die and will receive \$0.12 each time the sum of 3,4 or 5 occur, but will lose B when any other sum shows. Determine the value of B if the game is to be fair.

3,4,5	otherwise
$\frac{9}{36}$	$\frac{27}{36}$

$$.12\left(\frac{9}{36}\right) - B(27/36) = 0$$

$$.03 - 3/4\,B = 0 \qquad -3/4\,B = -.03 \qquad B = \$.04$$

Homework: Worksheet on Expected Value and Fairness