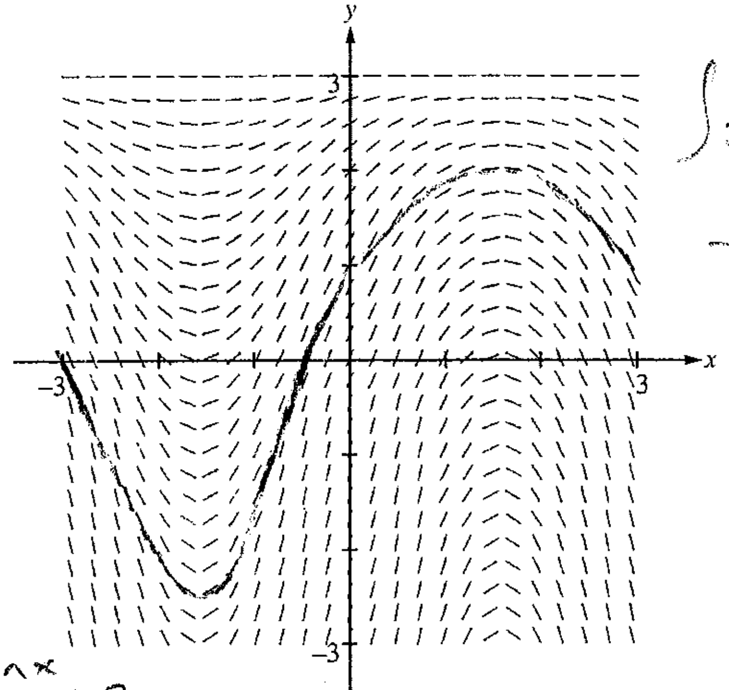
WORKSHEET

1 Consider the differential equation $\frac{dy}{dx} = (3 - y)\cos x$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(0) = 1$. The function f is defined for all real numbers.

(a) A portion of the slope field of the differential equation is given below. Sketch the solution curve through the point $(0, 1)$.



$$\int \frac{dy}{3-y} = \int \cos x \, dx$$

$$-\ln|3-y| = \sin x + C$$

$$\ln(3-y) = -\sin x + C$$

$$3-y = e^{-\sin x + C}$$

$$3-y = Ce^{-\sin x}$$

$$-y = Ce^{-\sin x} - 3$$

$$y = -Ce^{-\sin x} + 3$$

$$C = 2$$

$$y = -2e^{-\sin x} + 3$$

(b) Write an equation for the line tangent to the solution curve in part (a) at the point $(0, 1)$. Use the equation to approximate $f(0.2)$.

$\frac{dy}{dx} = 3-y \cos x$ $\quad \frac{dy}{dx} = (3-1)\cos 0$ $\quad \frac{dy}{dx} = 2$ $\quad y - 1 = 2(x - 0)$

$y = 2x + 1$

$y = 2(.2) + 1$

$= 1.4$

(c) Find $y = f(x)$, the particular solution to the differential equation with the initial condition $f(0) = 1$.

$$y = -2e^{-\sin x} + 3$$

$$y = -2e^{-\sin(.2)} + 3$$

$y = 1.36$ (actual)

2 At the beginning of 2010, a landfill contained 1400 tons of solid waste. The increasing function W models the total amount of solid waste stored at the landfill. Planners estimate that W will satisfy the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ for the next 20 years. W is measured in tons, and t is measured in years from the start of 2010.

(a) Use the line tangent to the graph of W at $t = 0$ to approximate the amount of solid waste that the landfill contains at the end of the first 3 months of 2010 (time $t = \frac{1}{4}$).

$y = 44t + 1400$

$\frac{1}{25}(1400 - 300) = \frac{1100}{25} = 44$

$y = 44(1/4) + 1400 = 1411$ tons

(b) Find $\frac{d^2W}{dt^2}$ in terms of W. Use $\frac{d^2W}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the amount of solid waste that the landfill contains at time $t = \frac{1}{4}$.

$\frac{d^2w}{dt^2} = \frac{1}{25}\frac{dw}{dt}$

$\frac{1}{625}(w - 300)$ $\quad w \geq 1400$

underestimate.

(c) Find the particular solution $W = W(t)$ to the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ with initial condition $W(0) = 1400$.

$$\int \frac{dw}{w-300} = \int \frac{1}{25}\, dt$$

$$-\ln|w-300| = \tfrac{1}{25}t + C$$

$$\ln|w-300| = -\tfrac{1}{25}t + C$$

$$w - 300 = Ce^{-1/25 t}$$

$$w = Ce^{-1/25 t} + 300$$

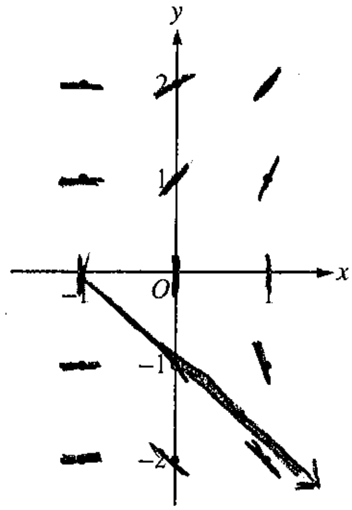
$$C = 1100$$

$$w = 1100e^{-1/25 t} + 300$$

3

Consider the differential equation $\dfrac{dy}{dx} = \dfrac{x+1}{y}$.

(a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated, and for $-1 < x < 1$, sketch the solution curve that passes through the point $(0, -1)$.



(b) While the slope field in part (a) is drawn at only twelve points, it is defined at every point in the xy-plane for which $y \neq 0$. Describe all points in the xy-plane, $y \neq 0$, for which $\dfrac{dy}{dx} = -1$.

$\frac{x+1}{y} = -1 \quad x+1 = -y$

$-x-1 = y \quad y \neq 0$

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = -2$.

$\int y\,dy = \int (x+1)\,dx \quad \frac{1}{2}y^2 = \frac{1}{2}x^2 + x + C \quad C = 2$

$2 = 0 + C \quad y^2 = x^2 + 2x + 4$

$y = -\sqrt{x^2+2x+4}$

4

The rate at which a baby bird gains weight is proportional to the difference between its adult weight and its current weight. At time $t = 0$, when the bird is first weighed, its weight is 20 grams. If $B(t)$ is the weight of the bird, in grams, at time t days after it is first weighed, then

$$\frac{dB}{dt} = \frac{1}{5}(100 - B).$$

$\frac{dB}{100-B} = \frac{1}{5}dt$

$-\ln|100-B| = \frac{1}{5}t + C$

$100 - B = Ce^{-1/5 t}$

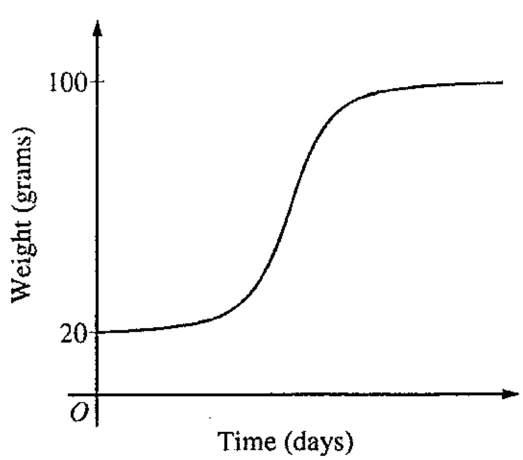
$C = 80$

Let $y = B(t)$ be the solution to the differential equation above with initial condition $B(0) = 20$.

(a) Is the bird gaining weight faster when it weighs 40 grams or when it weighs 70 grams? Explain your reasoning.

$\frac{dB}{dt} = \frac{1}{5}(60) = 12 \quad \frac{dB}{dt} = \frac{1}{5}(30) = 6$ @ 40 grams faster

(b) Find $\dfrac{d^2B}{dt^2}$ in terms of B. Use $\dfrac{d^2B}{dt^2}$ to explain why the graph of B cannot resemble the following graph.



$\frac{d^2B}{dt^2} = -\frac{1}{5}\frac{dB}{dt}$

$= -\frac{1}{5}\left(\frac{1}{5}(100-B)\right)$

$= \frac{-1}{25}(100-B)$

$-$ | $-$ |

20 100

concave down

(c) Use separation of variables to find $y = B(t)$, the particular solution to the differential equation with initial condition $B(0) = 20$.

$100 - B = 80e^{-1/5 t}$

$100 - 80e^{-1/5 t} = B$