Day 4: Concavity and the Second Derivative Test

Let f be differentiable on an open interval I. The graph of f is concave upward on I if $f'(x)$ is increasing on the interval and concave downward on I if $f'(x)$ is decreasing on the interval.

Let f be a function whose second derivative exists on an open interval I.

1. If $f''(x) > 0$ for all x in I, then the graph of f is concave upward in I.

2. If $f''(x) < 0$ for all x in I, then the graph of f is concave downward in I.

Points of Inflection: If $(c, f(c))$ is a point of inflection of the graph of f, then either $f''(c) = 0$ or $f''(c)$ does not exist at x = c.

I. For each of the following find all x coordinates of all points of inflection, find all discontinuities, and find all open intervals in which the function is concave up or concave down.

Ex: $f(x) = x^3 - 3x^2 + 4$

$f'(x) = 3x^2 - 6x$

$f''(x) = 6x - 6 \quad x = 1$

f'' $-$ $+$

1

$(-\infty, 1)$ concave downward

$(1, \infty)$ concave upward.

Ex: $f(x) = x^4 - 4x^3$

$f'(x) = 4x^3 - 12x^2$

$f''(x) = 12x^2 - 24x$

$12x(x-2) \quad x = 0 \quad x = 2$

f'' $+$ $-$ $+$ Pts $(0,0)$ $(2,-16)$

-1 0 1 2 4

CCU $(-\infty, 0) \cup (2, \infty)$

CCD $(0, 2)$

Ex: $f(x) = \dfrac{1}{x-3}$ $(x-3)^{-1}$

$f'(x) = -1(x-3)^{-2} = \dfrac{-1}{(x-3)^2}$

$f''(x) = 2(x-3)^{-3}$

f'' $-$ $+$ f'' DNE $x = 3$

3

CCU $(3, \infty)$

CCD $(-\infty, 3)$

Ex: $f(x) = \dfrac{x^2+1}{x^2-4}$

$f'(x) = \dfrac{(x^2-4)2x - (x^2+1)2x}{(x^2-4)^2} = \dfrac{-10x}{(x^2-4)^2}$

$f''(x) = \dfrac{10(3x^2+4)}{(x^2-4)^3}$ DNE $x = \pm 2$

-2 2

Ex: $f(x) = \cos(x)$ $(0, 2\pi)$

$f'(x) = -\sin x$

$f''(x) = -\cos x$ Pt of inflection $(\pi/2, 0)$ $(3\pi/2, 0)$

f'' $-$ $+$ $-$

0 $\pi/2$ $3\pi/2$ 2π

CCU $(\pi/2, 3\pi/2)$

CCD $(0, \pi/2) \cup (3\pi/2, 2\pi)$

Ex: $f(x) = 2x - \sin(x)$ $(0, 2\pi)$

$f'(x) = 2 - \cos(x)$

$f''(x) = +\sin x$

$x = \pi$ Pt of inflection $(\pi, 2\pi)$

$+$ $-$

0 $\pi/2$ π $3\pi/2$ 2π

CCU $(0, \pi)$

CCD $(\pi, 3\pi/2)$

What does $f''(x)$ tell us about $f'(x)$?

slope increasing

1. $f'(x) > 0$ and $f''(x) > 0$ A

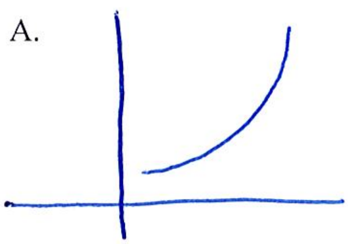
3. $f'(x) < 0$ and $f''(x) < 0$ C

2. $f'(x) < 0$ and $f''(x) > 0$ B

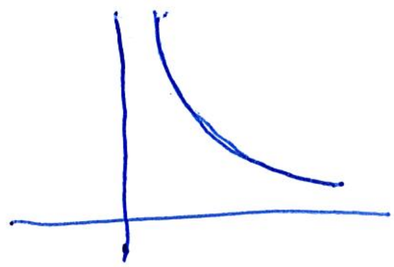
4. $f'(x) > 0$ and $f''(x) < 0$ D.

Match with the Graph?

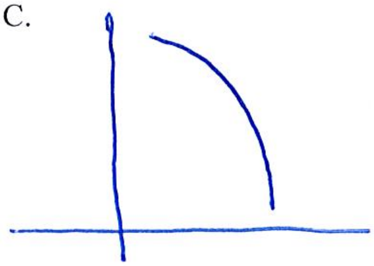
A.



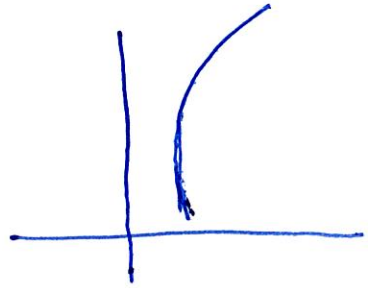
B.



C.



D.



Second Derivative Test:

Let f be a function such that $f'(x) = 0$ and the second derivative of f exists on an open interval containing c

1. If $f''(x) > 0$ then $f(c)$ is a relative minimum.

2. If $f''(x) < 0$ then $f(c)$ is a relative maximum.

If $f'(x) = 0$, the test fails. In such cases, you can use the First Derivative Test.

Ex: $f(x) = -3x^4 + 5x^3$

$f'(x) = -12x^3 + 15x^2$

$-3x^2(4x-5)$

$x=0 \quad x = 5/4$. critical #'s

$f''(x) = -36x^2 + 30x$

$-6x(6x-5)$

$f''(0) = 0$

$f''(5/4) < 0$

5/4 is a relative Minimum

Ex: $f(x) = 2x^4 - 8x + 3$

$f'(x) = 8x^3 - 8$

$= 8(x^3 - 1)$ $f''(1) > 0$

relative max

$f''(x) = 24x^2$ $x = 0$

(0,3) not a point of inflection.

$+ \quad | \quad +$

3

CCU $(-\infty, \infty)$